Improper integrals

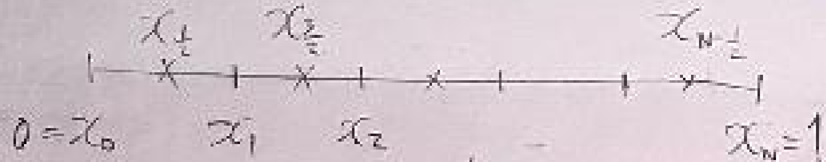
<u>Example</u> (*) $\int_0^1 x^p g(x)\,dx$

g: smooth $g(0) \neq 0$.

$p > -1$ (integrable)

	$-1<p<0$	$p>0$
Mid	✓	✓
Trap.	✗	✓
Simpson	✗	✓

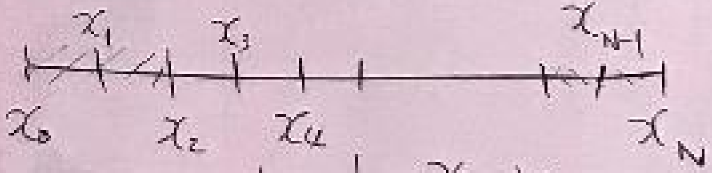
⇒ Error estimate (Midpoint Rule)



$$x_j - x_{j-1} = h$$

$$I_h = h\sum_{j=1}^{N} f(x_{j-\frac{1}{2}}) \quad \cdots \underline{\underline{(M)}}$$

Alternative



$$h = x_j - x_{j-1}$$

$$I_h = 2h\sum_{j=1}^{\frac{N}{2}} f(x_{2j-1}) \quad \cdots (M')$$

For (M)

$$I_h - I = \frac{-h^3}{12} \sum_{j=1}^{N} f''(\xi_j)$$

$$\cong \frac{-h^2}{12} \int_0^1 f''(\xi) d\xi$$

<u>Case 1</u>. $f''(\xi)$ integrable in $\int_0^1$

(i.e. $\underline{\underline{p > 1}}$ in $(*)$)

$$I_h - I = O(h^2)$$

<u>Case 2</u> $-1 < p < 1$ $(p \neq 0)$

$$I_h - I = \frac{-h^2}{12}\left(h \sum_{j=1}^{N} f''(\xi_j)\right)$$

$$= \frac{-h^2}{12}\left(h \sum_{j=1}^{1} + h \sum_{j=2}^{N}\right)$$

$$h \sum_{j=2}^{N} \cong \int_{x_1}^{x_N} f''(\xi) d\xi$$

$$= \int_{h}^{1} f''(\xi) d\xi = O(h^{p-1})$$

$$\frac{-h^2}{12} h \sum_{j=1}^{1} = (I_h - I)_{[0,h]} = h f\left(\tfrac{h}{2}\right) - \int_0^h f(x) dx$$

$$= h O\left(\tfrac{h}{2}\right)^p - O(h^{p+1})$$

$$\therefore I_h - I = O(h^{p+1})$$

Errors

	$-1<p<0$	$0<p<1$	$1<p<3$	$p>3$
M	$O(h^{p+1})$	$O(h^{p+1})$	$O(h^2)$	$O(h^2)$
T.	X	$O(h^{p+1})$	$O(h^2)$	$O(h^2)$
S.	X	$O(h^{p+1})$	$O(h^{p+1})$	$O(h^4)$

Desingularizization. $\int_0^1 \frac{f(x)}{x^\xi}\,dx, \quad 0<\xi<1$

Method 1. $f(x) = f(x) - P_k(x) + P_k(x)$

where $P_k(x)$ is Taylor polynomial of f near the singularity.

So that $\frac{f - P_k(x)}{x^\xi} \in C^k[0,1]$

enough to apply M. T. or S
$(k=2) \quad (k=4)$

<1

Method 2: change of variables

Example: $\int_0^1 \frac{e^x}{x^\gamma}\,dx \quad 0<\gamma<1$

$x = t^\alpha \qquad x^\gamma = t^{\alpha\gamma}$

$dx = \alpha t^{\alpha-1}\,dt$

$$= \int_0^1 \left(1 + t^\alpha + \frac{t^{2\alpha}}{2!} + \frac{t^{3\alpha}}{3!} \cdots\right) \alpha t^{\alpha(1-\gamma)-1}$$

Case 1: $\gamma = \frac{n}{m}, \; m, n \in \mathbb{N}$

$$\underset{\alpha=m}{\overset{\text{take}}{=}} \int_0^1 \underbrace{\left(1 + t^m + \frac{t^{2m}}{2!} + \cdots\right) m t^{m-n-1}}_{C^\infty[0,1]} dt$$

Case 2: ξ is irrational.

$$= \alpha \int_0^1 t^{\alpha(1-\xi)-1} + \frac{t^{\alpha(2-\xi)-1}}{2!} + \cdots$$

Need $t^{\alpha(1-\xi)-1} \in C^k$

$$\begin{cases} k = 2 \text{ for T.M.} \\ k = 4 \text{ for S.} \end{cases}$$

$$\alpha(1-\xi)-1 \geq k$$

$$\alpha \geq \frac{k+1}{1-\xi}$$

Example. $\int_1^\infty x^{-\frac{3}{2}} \sin(\frac{1}{x}) \, dx$

$x = t^{-\beta}$, $\beta > 0$

$dx = -\beta t^{-\beta-1}$

$$= -\int_{t=1}^{t=0} t^{\frac{3\beta}{2}} \sin(t^\beta) \beta t^{-\beta-1} \, dt$$

$$= \beta \int_{t=0}^{1} t^{(\frac{\beta}{2}-1)} \sin(t^\beta) \, dt$$

take $\beta = 2$ ($\because x^{-\frac{3}{2}}$, $n = 2$)

$$= 2 \int_0^1 \sin(t^2) \, dt$$

Example: $\displaystyle\int_0^\infty \frac{1}{1+x^4}\,dx$

$x \in [0,\infty)$, $\quad x+1 \in [1,\infty)$

$x+1 = t^{-\beta}$ $\quad$ ($\beta = 1$ will do)

$x = \frac{1}{t} - 1$, $\quad dx = -t^{-2}\,dt$

$$\int_{\substack{x=0\\(t=1)}}^{\substack{x=\infty\\(t=0)}} \frac{1}{1+\left(\frac{1}{t}-1\right)^4}\left(\frac{-1}{t^2}\,dt\right)$$

$$= \int_{t=0}^{1} \frac{t^2}{t^4+(1-t)^4}\,dt$$