Want to Solve $f(x)=0$,
$x, f \in \mathbb{R}^n$
Fixed Point Iteration:
Put in equivalent form

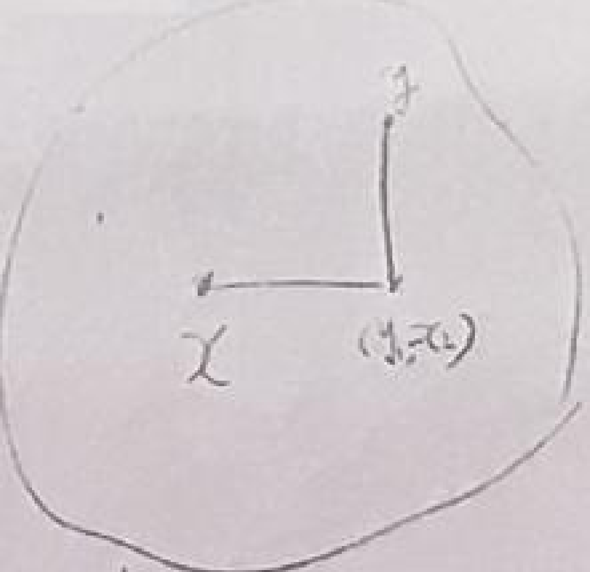
$$x = G(x) \Leftrightarrow f(x) = 0$$

and perform

$$x^{k+1} = G(x^k)$$

$$\begin{array}{ccc} x^{k+1} & = & G(x^k) \\ \downarrow & & \downarrow \text{ If } G \text{ is continuous} \\ x^* & & G(x^*) \end{array}$$

Proof of Theorem 10.4 under additional assumption that partial derivatives of f are continuous in D:



$$|f(x_1,x_2)-f(y_1,y_2)|$$

$$\le |f(x_1,x_2)-f(y_1,x_2)|+|f(y_1,x_2)-f(y_1,y_2)|$$

$$= |\partial f(\xi_1,x_2)||y_1-x_1|+|\partial_2 f(y_1,\xi_2)||y_2-x_2|$$

Fixed Point Iteration

$$X^{k+1} = G(x^k)$$

$$x^* = G(x^*)$$

$$x^{k+1} - x^* = G(x^k) - G(x^*)$$

$$H(t) = G(x^* + t(x - x^*))$$

$$G(x^k) - G(x^*) = H(1) - H(0) = \int_0^1 \frac{d}{dt} H(t)\, dt$$

$$= \underbrace{\int_0^1 G'(x_* + t(x^k - x^*))\, dt}_{\equiv \bar{G}'} \cdot (x^k - x^*)$$

$$x^{k+1} - x^* = \bar{G}'$$

$$\| x^{k+1} - x^* \|_p \leq \|$$

where $\|$

$$x^{k+1} - x^* = \bar{G}' (x^k - x^*)$$

$$\| x^{k+1} - x^* \|_p \leq ||| \bar{G}' |||_p \, \| x^k - x^* \|_p$$

where $||| A |||_p = \sup_{y \in \mathbb{R}^n \setminus 0} \dfrac{\| A y \|_p}{\| y \|_p}$

Note. ① $\left||| \int_0^1 G'(\cdots) dt \right||| \leq \int_0^1 ||| G'(\cdot\, t) ||| \, dt$

② If $\left| \dfrac{\partial g_i}{\partial x_j} \right| \leq \dfrac{\alpha}{n}$, $G = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$

$\Rightarrow ||| \bar{G}' |||_\infty \leq \alpha$

What if $x^{k+1} = G(x^k)$ diverges?

Method 1:

$$x^{k+1} = \alpha x^k + (I-\alpha) G(x^k)$$

Find suitable α ($n \times n$ real matrix)

$$x_* = \alpha x_* + (I-\alpha) G(x_*)$$

$$e^{k+1} = \alpha e^k + (I-\alpha) \bar{G}' e^k$$

$$= (\alpha + (I-\alpha) \bar{G}') e^k$$

Ideal $\alpha = \alpha^*$: $\alpha^* + (I-\alpha^*) \bar{G}' = 0$

$$\alpha^* = (\bar{G}'-I)^{-1} \bar{G}'$$

Method 2:

$$f(x) = 0$$

$$f_1(x) + f_2(x) = 0$$

where f_1 is linear ($f_1(x) = Ax$) and $O(1)$

A is a fixed $n \times n$ matrix

f_2 is small

$$x = A^{-1}(f_2(x))$$

$$x^{k+1} = A^{-1}(f_2(x^k))$$

$$e^{k+1} = A^{-1} f_2' e^k$$