Homework Assignment for Week 15

1. Section 6.6: Problems 3(a,c), 5(a,c), 14(a,b), 17, 24, 25, 32.

2. Derive a system of equation corresponding to the following boundary value problem

$$\begin{aligned} &u''(x) - u(x) = f(x), \quad x \in (0,1) \\ &u(0) = \alpha,\ u(1) = \beta \end{aligned}$$

   with uniformly spaced grids $0 = x_0 < x_1 < \cdots < x_N = 1$, $x_i - x_{i-1} = h = 1/N$, using second order finite difference method. That is, given α, β, and f_i, $i = 1, 2, \cdots, N-1$, try to derive a linear system of equations to solve for u_i, $i = 1, 2, \cdots, N-1$.

3. Implement your own tridiagonal solver for previous problem using indexing of the entries as in Section 6.6, Problem 29 (i.e. only store a_i, b_i and c_i, not the whole matrix with a lot of zero entries). For this problem, pivoting is not needed. Take $\alpha = 1$, $\beta = e$ and $f(x) = 0$. The exact solution is $u^e(x) = e^x$. Check numerically that indeed $\max_i |u^e(x_i) - u_i^h| = O(h^2)$ by taking $N = 100$ and $N = 200$.

4. Let A be the matrix resulted from discretizing the Poisson equation in a 2D cell,

$$\begin{aligned} &(\partial_x^2 + \partial_y^2)u(x,y) = f(x,y), \qquad (x,y) \in (0,1)^2 \\ &u = 0, \quad \text{on the boundary of } (0,1)^2 \end{aligned} \tag{1}$$

   with uniformly spaced grids $0 = x_0 < x_1 < \cdots < x_N = 1$, $0 = y_0 < y_1 < \cdots < y_N = 1$, $x_i - x_{i-1} = y_j - y_{j-1} = h = 1/N$, using second order centered finite difference method. The unknowns are u_{ij} with with $1 \le i, j \le N-1$. Give the leading order operation KN^p for for LU decomposition and forward, backward substitution.

   Repeat the procedure for the 3D case.

5. Section 7.3: Problems 7(a), 8(a).