Homework Assignment for Week 09

1. Section 3.5:

   The cubic spline with not-a-knot end condition gives rise to a linear system $Ax = b$ where A is an $(n+1)\times(n+1)$ matrix and $x = (c_0, c_1, \cdots, c_n)^T$. Eliminating c_0 and c_n results in a new equivalent linear system $\tilde{A}\tilde{x} = \tilde{b}$ where $\tilde{x} = (c_1, \cdots, c_{n-1})^T$ and $\tilde{A}$ can be made symmetric. Carry out the details for the case $h_0 = \cdots = h_n = h$ and show that $\tilde{A}$ is symmetric and positive definite (hence the original A is non-singular).

   Hint: Under not-a-knot end condition, $S''(x)$ is a linear function on $[x_0, x_2]$ and on $[x_{n-2}, x_n]$ (why?). This leads to an linear equation relating c_0, c_1 and c_2, and one relating c_{n-2}, c_{n-1} and c_n.

2. Section 3.5:

   Find the rate of convergence (as Ch^p, where $h = x_j - x_{j-1}$) of cubic spline interpolation numerically with default (not-a-knot) and clamped boundary conditions by evaluating roughly 10 points on each interval $[x_{j-1}, x_j]$.

   For example, let (x_j, y_j), $j = 0, \cdots, n$ be data points where $x_j = a + jh$, $h = (b-a)/n$ and $y_j = f(x_j)$. Denote by $x_k^* = a + kh^*$, $h^* = h/10$, $k = 0, \cdots, 10n$ the points to be interpolated. Take $a = 1, b = 5$, $f(x) = 1/x$, $n = 16, 32, \cdots$ and evaluate $\max_k |f(x_k) - S(x_k)|$ to find the order of accuracy p numerically.

   Optional: Do the same for natural cubic spline. Key words: csape (advanced version of spline in MATLAB), 'variational'.

3. Section 4.1: Problems 24, 26, 28, 29.

   Definition: The linear combination $\sum_{i=-k}^{k} c_i f(x_0 + ih)$ is a pth order approximation of $f^{(q)}(x_0)$ if

   $$\sum_{i=-k}^{k} c_i f(x_0 + ih) = f^{(q)}(x_0) + O(h^p)$$

4. Section 4.1:

   Apply the round-off error instability calculation (end of section 4.1) to second order approximation of $f''(x_0)$. Find the critical h^* that minimizes $e(h)$. Express both h^* and $e(h^*)$ in terms of the machine ε as $O(\varepsilon^p)$ and find p for both h^* and $e(h^*)$.

5. Section 4.1: Repeat previous problem for fourth order approximation of $f'(x_0)$.

6. Section 4.1, 4.2:

   Derive the five-point formula for $f'''(x_0)$ and and $f^{(4)}(x_0)$, respectively using $f(x_0)$, $f(x_0 \pm h)$ and $f(x_0 \pm 2h)$. You can either use the method of undetermined coefficients (combined with Taylor expansion around x_0), or repeat the three-point formula for $f'(x_0)$ and/or $f''(x_0)$ and the identities $f''' = (f')'' = (f'')'$ and $f'''' = (f'')''$ to get the five-point formulae for $f'''(x_0)$ and and $f^{(4)}(x_0)$.