Homework Assignment for Week 02

Goal: Review basic programming skills. Understand stability, instability and rate of convergence.

0. Hint for problem 22, section 1,2: Would '≈' be replaced by '=' if there were no numerical errors in $\sum_{i=0}^{9}(\cdots)$ at all (and sum remains from $i = 0$ to 9)?

1. Section 1.3: Problems 4, 7(c), 10, 15.

2. Consider the following recursive relation

$$x_1 = a_1, \quad x_2 = a_2, \quad x_n = \frac{9}{2}\,x_{n-1} - 2\,x_{n-2}$$

   (a) What is the exact solution of x_n with $a_1 = 1/5$, $a_2 = 1/10$? Is it stable (in relative error)? Verify your answer numerically. for n around 40 or larger.

   (b) Do the same for $a_1 = a_2 = 1/5$ and for $a_1 = 1, a_2 = 1/2$, respectively. Explain your observation.

   (c) Which of the above, and the example of page 32-33 of the textbook (note in this example $a_0 = 1$, $a_1 = 1/3$, so $a_2 = 1/9$), is(are) stable when you compute forward to a_n, and then backward to a_1 (using computed a_n and a_{n-1}) recursively? Why?

3. Read the details about 'loglog', 'semilogx', 'semilogy' in matlab/octave. Typical convergence behavior, such as $y_n = C_1 n^{-k}$, or $z_n = C_2\alpha^n$, where n denotes the number of iterations and $C_1 > 0$, $C_2 > 0$, $k > 0$ and $0 < \alpha < 1$ are some constants, will have distinct behaviors when you choose the correct scaling. That is, if you try to plot y_n or z_n versus n in one of the special scalings above, you will see a straight line. You can further find the slope of the straight line by plotting together with a known sequence (such as $a_n = 1/n$ versus n) and check if the two lines are parallel.

   (a) Try to analyze and find the rate of convergence of

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i^2} = \frac{\pi^2}{6}$$

   numerically by plotting the results in the correct $x-$ and $y-$ scaling.

   (b) Assuming that the leading term of the error is of the form

$$\sum_{i=1}^{n} \frac{1}{i^2} - \text{limit} \approx C n^{-p}$$

   An alternative method to estimate the convergence rate (with or without using the information that the limit is $\frac{\pi^2}{6}$) is to estimate p using the numerical values of $\sum_{i=1}^{N}$ with two or three different values of N, say $N = 100$, 200 and 400. Find p with this method.

4. Section 2.1: Problems 14. Write your own code to practice various stopping conditions such as equation (2.1)-(2.3).

5. Study the 'while' loop in your programming language (preferably C, C++ or popular ones in the C-family) and write a program that, given any real number a, finds the largest N such that

$$\sum_{i=1}^{N} \frac{1}{i} < a.$$

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