# Final Exam

Jan 12, 2018.

1. (15 pts) Use any desingularization method to evaluate the improper integral

$$\int_0^\infty \frac{1}{1+x^3}\,dx$$

with a composite quadrature rule of 4th order accuracy. Demonstrate numerically that the result is indeed 4th order accurate.

2. (15 pts) Consider discretizing

$$\begin{aligned}&(\partial_x^2+\partial_y^2)u(x,y)=f(x,y), \qquad (x,y)\in(0,5)\times(0,1)\\ &u=0, \quad \text{on the boundary of } (0,5)\times(0,1)\end{aligned}$$

with uniformly spaced grids $0=x_0<x_1<\cdots<x_{5N}=5$, $0=y_0<y_1<\cdots<y_N=1$, $x_i-x_{i-1}=y_j-y_{j-1}=h=\frac{1}{N}$, using second order centered finite difference method. The following two possible ordering of the unknowns $u_{i,j}$,

$$(i,j)=(1,1)\to(1,2)\to\cdots\to(1,N-1)\to(2,1)\to(2,2)\to\cdots, \tag{1}$$

and

$$(i,j)=(1,1)\to(2,1)\to\cdots\to(5N-1,1)\to(1,2)\to(2,2)\to\cdots, \tag{2}$$

result in two linear systems $A^{(1)}u^{(1)}=f^{(1)}$ and $A^{(2)}u^{(2)}=f^{(2)}$. For large $N$, which linear system is faster to solve using Gaussian elimination without pivoting, combined with backward substitution? or are they the same? Explain.

3. (7+7 pts)

(a) Perform the required row interchanges for the following linear system using scaled partial pivoting:

$$\begin{array}{rrrl} & x_2+ & x_3= & 4\\ x_1- & 2x_2- & x_3= & 5\ ,\\ x_1- & x_2+ & x_3= & 6\end{array} \tag{3}$$

Give all details and the final linear system. Need not solve it.

(b) Find $P$, $L$, $U$ in the factorization $PA=LU$ for the matrix

$$A=\begin{pmatrix} 0 & 0 & -1 & 1\\ -1 & -1 & 2 & 0\\ 1 & 1 & -1 & 2\\ 1 & 2 & 0 & 2\end{pmatrix} \tag{4}$$

where $P$ is the permutation matrix corresponding to partial pivoting.

4. (15 pts) True or False? Explain.

   Suppose that $A = M - N$ is nonsingular. Then the iteration $Mx^{(k+1)} = b + Nx^{(k)}$ converges for any $x^{(0)}$ if $M$ is nonsingular and $T = M^{-1}N$ satisfies $\|T\| < 1$ for some matrix norm $\|\cdot\|$.

5. (15 pts) Write down the definition of 'condition number'. Derive a related inequality that gives an estimate of relative error of the solution of a linear system.

6. (15 pts) Consider the integral equation

$$u(x_i) = f(x_i) + \int_0^1 K(x_i, t)\, u(t)\, dt, \quad i = 0, \cdots, M, \tag{5}$$

   where $0 = x_0 < x_1 < \cdots < x_M = 1$, with $x_j - x_{j-1} = h = \frac{1}{M}$, $f(x) = x^2$, $K(x,t) = -\frac{1}{3}e^{-|x-t|}$ and $\int_0^1 dt$ is evaluated using trapezoidal rule.

   Form the linear system, find an convergent iterative method, explain why your method converges. Need not solve it.

7. (15 pts) Find the polynomial $p(x)$ that minimizes $\int_0^1 (\sin x - p(x))^2 dx$ among all odd polynomials ($p(-x) = -p(x)$) with degree less than or equal to 5. Derive the linear system and need not solve it numerically.

8. (8 pts) Find the constants $a, b, c$ so that, the quadrature $\displaystyle\int_{-h}^{h} f(x)dx \approx h \cdot (af(-h) + bf(0) + cf(h))$ has highest degree of precision, where $f$ is sufficiently smooth. Under the assumption (need not prove this assumption) that the error of this quadrature is of the form

$$\int_{-h}^{h} f(x)dx - h \cdot (af(-h) + bf(0) + cf(h)) = Kf^{(n)}(\xi)h^p,$$

   find $K$, $n$ and $p$.

9. (8 pts) Use any method to solve the nonlinear system of equations

$$\begin{aligned} 1x_1 + 2x_2 + 0.03 * \sin(x_1 + x_2) &= 4 \\ 5x_1 + 6x_2 + 0.07 * \cos(x_1 - x_2) &= 8 \end{aligned}$$

   Write your answer in the format of 'format long e' and hand in your code.