Midterm 02

Dec 02, 2011, 13:10-16:30.

As before, estimate to leading order means CN^p, find C and p. Unless otherwise specified, L in LU means $l_{ii} = 1$.

1. (a) Write a pseudo-code for Gauss elimination (without pivoting) and estimate number of multiplication/division needed to leading order for a dense $N \times N$ matrix A.
   (b) Consider an $N^2 \times N^2$ matrix B with $B_{ij} = 0$ except for $i - j = 0, 1, \pm N$ (only 4 diagonals have nonzero entries). Estimate total number of multiplication needed for Gaussian elimination (elimination part only) on B to leading order. Explain.
   (c) Consider an $N^2 \times N^2$ matrix C with $c_{ij} = 0$ except for $i - j = 0, \pm N$ (only 3 diagonals have nonzero entries). Estimate total number of multiplication needed for Gaussian elimination (elimination part only) on C to leading order. Explain.

2. Which of the following matrices admit LU decomposition (that is, no 'P' needed)? Then find L and U (use paper and pencil) for those matrices. Explain.

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}$$

3. True or false? Prove or disprove.

   Denote by A_k the $k \times k$ the leading principles of an $N \times N$ real matrix A. If $\det A_k \neq 0$, $k = 1, \cdots, N$. Then A admits an LU decomposition.

4. Let A be a symmetric positive definite $N \times N$ matrix. Suppose that Gaussian elimination on A does not require pivoting. Prove that A admits Choleski decomposition. Suppose further that $a_{ij} = 0$ except for $i - j = 0, \pm 1, \pm 2$. Write a pseudo-code for it and find the number of multiplications/divisions needed to leading order, assuming square root amounts to 10 multiplications

5. Let $A =$ be a sparse matrix and $A = D - L - U$ with D diagonal, L strictly lower triangular and U strictly upper triangular ($l_{ii} = 0 = u_{ii}$).
   (a) Write down the Jacobi, Gauss-Siedel and SOR iteration for solving $Ax = b$, respectively.
   (b) Suppose T_j is diagonalizable and $\rho(T_j)$, the largest absolute value of eigenvalues of T_j, $= 0.94$, and initial error $\|x^{(0)} - x\| = 1$. Find the number of Jacobi iteration it takes to reach $\|x^{(k)} - x\| \leq 10^{-4}$.
   (c) Suppose that, for some sparse matrix B, $\rho(T_j)$ is diagonalizable with real eigenvalues $2 \leq \lambda(T_j) \leq 3$. Propose a convergent (and fastest, if possible) iterative method for solving $By = c$. Explain.

6. Suppose it is known that a nonsingular tridiagonal matrix B admits the decomposition $B = \tilde{U}\tilde{L}$ where $\tilde{L}$ is lower triangular and $\tilde{U}$ upper diagonal with $\tilde{u}_{ii} = 1$. Show that such a decomposition is unique and write a pseudo-code for it. Then apply it to the 10×10 matrix B with $b_{ii} = 6.1$, $b_{i+1,i} = 1 = b_{i,i+1}$ and $b_{ij} = 0$ otherwise. Copy your code and report $\sum_{i=1}^{10} \text{abs}(\tilde{l}_{ii})$. If you can't do it, do the LU case with partial credits.