# Midterm 02

Dec 03, 2010. Total 108 points, 12 points each.

1. True or False?

   Let $P_n$ be the degree $n$ interpolating polynomial of $\cos x$ with uniformly spaced nodes on $[0, 1]$. Then
   $$\max_{0\le x\le 1} |\cos x - P_n(x)| \to 0 \quad \text{as } n \to \infty$$

2. For a function $f$, the Newton divided difference formula gives the interpolating polynomial
   $$P_3(x) = 1 + 4x + 4x(x - 0.25) + \frac{16}{3}x(x - 0.25)(x - 0.5),$$
   on the nodes $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$ and $x_3 = 0.75$. Find $f[x_1, x_2, x_3]$.

3. Let $i_0, i_1, \cdots, i_n$ be a rearrangement of the integers $0, 1, \cdots, n$. Show that
   $$f[x_0, x_1, \cdots, x_n] = f[x_{i_0}, x_{i_1}, \cdots, x_{i_n}]$$

4. What is the degree of the Hermite polynomial for the data $(x_0, f(x_0), f'(x_0)), (x_1, f(x_1), f'(x_1)), \cdots, (x_n, f(x_n), f'(x_n))$? Prove that the interpolating polynomial of this degree is unique.

5. Let $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. Find a polynomial $P(x)$ such that $P(x_0) = P'(x_0) = 0$, $P(x_2) = P'(x_2) = 0$ and $P(x_1) = 1$, $P'(x_1) = 2$, $P''(x_1) = 3$. (Note this problem is different from the one in quiz 4). You can start by analyzing the degree needed for this polynomial. Need not simplify your answer.

6. Suppose that we are to construct a piecewise polynomial interpolation $S(x)$ on the data $(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$, with additional continuity conditions for $S'$, $S'$, $S''$ and $S'''$ on the interior nodes $x_1, \cdots, x_{n-1}$. What is the minimal degree needed in each interval? How many additional end conditions are needed? Explain.

7. Interpolate the function $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$ with equally spaced nodes $-5, -4, \cdots, 4, 5$ using the built-in spline function with 'clamped' boundary condition in octave. Give the interpolated value at $x = 0.1$ with 10 digits. The answer should be close to $f(0.1)$. You can write your code to the answer sheet for potential partial credit in case the answer is wrong. Note again this is a different problem from the one in quiz 4.

8. Derive a 4th order centered difference approximation for $f'(x)$ and $f''(x)$ respectively. You can use any method. For example, undetermined coefficient or Richardson extrapolation, or others.

9. Suppose a scheme gives numerical approximations $N(0.2) = 1.2516$, $N(0.1) = 1.2327$ and $N(0.05) = 1.2303$ for the true value $M$. What is the order of this numerical scheme? Can you give a better approximation of $M$ than $N(0.05)$?