Midterm 01

Oct 29, 2010.

1. (8 pts) How many "bits" does it take to store floating point numbers of the form $\pm 1.a_1a_2\cdots a_s \times 2^e$ with $s = 23$, $a_j \in \{0,1\}$, $-127 \le e \le 128$? What is the largest number of this form?

2. (10 pts) Evaluate
$$p(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!}$$
as efficient as possible. You may give your answer either in the form of a loop, or any other expression as long as it is clear enough. How many multiplications are needed?

3. (10 pts) Solve for $x^2 - 1900x + 1 = 0$ to 15 correct digits. Every digit counts. Explain how you find your answer. (direct evaluation using 'calculator' will receive no credits).

4. (10 pts) Is the following algorithm stable or not? $p_0 = 1, \quad p_1 = 1/3, \quad p_n = \frac{10}{3}p_{n-1} - p_{n-2}$. Explain (with mathematical reasoning, not numerical observation). The true solution is $p_n^{\mathrm{e}} = (\frac{1}{3})^n$.

5. (8+4 pts)

   (a) Give a convergent fixed point iteration for solving $f(x) = x + 2\sin(x) - 0.01$. Just give the formulae, no numerical values needed.

   (b) Give an upper bound for the number of steps it takes to reach $|x_n - x^*| < 10^{-6}$ with $x_0 = 1$.

6. (6+8+6 pts) Suppose that $g : \mathbb{R} \mapsto \mathbb{R}$, $g \in C^2(\mathbb{R})$ and $|g'(x)| < 1/2$.

   (a) Prove that the equation $x = g(x)$ has a unique solution.

   (b) Prove (in detail) that the iteration $x_{n+1} = g(x_n)$ always converges to the solution.

   (c) Give formulae of Steffensen's method for this problem.

7. (4+8 pts)

   (a) Give the formula of Newton's method for solving $x^2 - 2x + 1 = 0$.

   (b) Find the order of convergence and prove your answer.

8. (4+8 pts)

   (a) Apply Aitken's Δ^2 method to the sequence $p_n = \cos\frac{1}{n}$.

   (b) Find $\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p}$.

9. (6 pts) Suppose Müller's method is applied to locate a root of $f(x) = x^3 - 2 = 0$, with $x_0 = 1$, $x_1 = 2$ and $x_2 = 3$. What is x_3?