Quiz 05

Dec 22, 2017.

1. Use any desingularization method to evaluate the improper integral

$$\int_0^1 x^{\frac{-1}{3}} e^x \, dx$$

with a composite quadrature rule of 4th order accuracy. Demonstrate numerically that the result is indeed 4th order accurate.

Answer:
Method 1
Apply desingularization method to recover the 4th order accuracy of Composite Simpson or Composite Gaussian:

$$\int_0^1 x^{-1/3} e^x \, dx = \int_0^1 x^{-1/3}(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}) \, dx + \int_0^1 x^{-1/3} \cdot (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}) \, dx \; \textbf{(4 pts)}$$

Check the error order numerically:

$$\log_2 \left| \frac{I_h(100) - I_h(200)}{I_h(200) - I_h(400)} \right| = 4.00\ldots \; \textbf{(16 pts)}$$

Write C **(extra 4 pts)**

Method 2
Use change of variable $x = y^p$:

$$\int_0^1 x^{-1/3} e^x \, dx = p \int_0^1 e^{y^p} y^{\frac{2}{3}p - 1} \, dy$$

Find proper $p > 0$ such that the intergrand is C^4. **(4 pts)**

Check the error order numerically **(16 pts)**

Write C **(extra 4 pts)**

2. Let A be an $N^2 \times N^2$ matrix with $a_{ij} = 0$ except for $-N \le i - j \le 2N$ (only $3N + 1$ diagonals have nonzero entries). Suppose that $Ax = b$ can be solved using Gaussian elimination without pivoting. Find the leading order operation count (multiplication/division only) CN^p for the elimination and backward substitution, respectively. Give all details.

Answer:
GE: details **(5 pts)**, $2N^4$ **(5 pts)**
BS: details **(5 pts)**, N^3 **(5 pts)**

3. Write a pseudo-code for Gaussian elimination, assuming no pivoting is needed. Then another pseudo-code for backward substitution. Partial credits is nearly impossible for incorrect algorithms. Check carefully.

   Answer: Refer to Algorithm 6.1 in the textbook.
   GE: **(10 pts)**
   BS: **(10 pts)**

4. Perform the required row interchanges for the following linear system using scaled partial pivoting:

$$\begin{array}{rrrl} & x_2+ & x_3 = & 6 \\ x_1- & 2x_2- & x_3 = & 4 \\ x_1- & x_2+ & x_3 = & 5 \end{array}, \tag{1}$$

   Give all details and the final linear system. Need not solve it.

   Answer:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 1 & -2 & -1 & 4 \\ 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{E_1 \leftrightarrow E_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & -2 & -1 & 4 \\ 0 & 1 & 1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{E_2 \leftrightarrow E_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & 6 \\ 0 & -1 & -2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -1 & 5 \end{array}\right]$$

   (5 pts for each step)

5. Find P, L, U in the factorization $PA = LU$ for the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \tag{2}$$

   where P is the permutation matrix corresponding to partial pivoting.

   Answer:
   Details of finding U: **(6 pts)**

$$U = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \text{ \textbf{(2 pts)}}$$

   Details of finding P: **(4 pts)**

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ \textbf{(2 pts)}}$$

Details of finding L: **(4 pts)**

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \textbf{(2 pts)}$$