Quiz 04

Nov 24, 2017.

1. Give an approximation, $f_h''(x)$, of of $f''(x)$ from $f(x-h)$, $f(x)$ and $f(x+h)$. Then derive an error identity of the form $f''(x) - f_h''(x) = C_1 f^{(?)}(\xi)h^?$.

   Ans: Refer to the textbook for the derivation of error term. **(12 pts)**

   $$f_h''(x) = \frac{1}{h^2}[f(x-h) - 2f(x) + f(x+h)] \textbf{ (4 pts)}$$

   $$f''(x) - f_h''(x) = C_1 f^{(4)}(\xi)h^2 \textbf{ (4 pts)}$$

2. Find $\min_{h>0} e(h) = \min_{h>0} |f''(x) - f_h''(x)|$ where $f_h''(x)$ is the three point formula from previous problem. Express the critical value h^* and the minimum $e(h^*)$ in terms of machine ε as $O(\varepsilon^\alpha)$ and find α for them.

   Ans: The round-off error is bounded by

   $$\frac{4\epsilon}{h^2} + \frac{h^2}{12}M$$

   where M is an upper bound of $|f^{(4)}|$. **(12 pts)**
   Thus, the optimal $h^* = O(\epsilon^{1/4})$ **(4 pts)** and $e(h^*) = O(\epsilon^{1/2})$ **(4 pts)**.

3. Derive a fourth order approximation of $f'(x)$ from $f(x)$, $f(x \pm h)$, $f(x \pm 2h)$, $f(x \pm 3h), \cdots$. Assume $f \in C^\infty$ and show that your formula satisfies $|f'(x) - f_h'(x)| \le Ch^4$.

   Ans: Refer to the textbook for the derivation of approximation. **(12 pts)**

   $$f_h'(x) = \frac{1}{12h}[f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)] \textbf{ (4 pts)}$$

   Show the inequality of error term. **(4 pts)**

4. Let $x_0 = a$, $x_1 = \frac{a+b}{2}$, and $x_2 = b$. Write down trapezoidal rule, the midpoint rule and Simpson's rule approximations of $\int_a^b f(x)dx$. Then derive the error formula (equality) for any one of them of your choice.

   Ans:
   Refer to the textbook for the derivation of error term **(4 pts)**. Error term **(4 pts)**.
   Trapezoidal: $\frac{h}{2}[f(x_0) + f(x_2)]$ where $h = b - a$ **(4 pts)**. Error: $-\frac{h^3}{12}f''(\xi)$.
   Midpoint: $2hf(x_1)$ where $h = (b-a)/2$ **(4 pts)**. Error: $\frac{h^3}{3}f''(\xi)$.
   Simpson's: $\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$ where $h = (b-a)/2$ **(4 pts)**. Error: $-\frac{h^5}{90}f^{(4)}(\xi)$.

5. Evaluate the degree of precision for all three methods in previous problem. Give definition and show details. You may choose any a, b for your convenience (-1, 1 are recommended).

   Ans:
   Definition of degree of precision. **(5 pts)**
   Trapezoidal: DOP=1, details. **(5 pts)**
   Midpoint: DOP=1, details. **(5 pts)**
   Simpson's: DOP=3, details. **(5 pts)**

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