# Quiz 03

Nov 10, 2017.

1. Let $P_n$ be the degree $n$ interpolating polynomial of $\cos(2x)$ on the uniformly spaced nodes $x_0, \cdots, x_n$ on $[0,1]$ with $x_j = jh$, $h = 1/n$. Is it true that

$$\max_{0\le x\le 1} |\cos(2x) - P_n(x)| \to 0 \quad \text{as } n \to \infty?$$

Explain.

**Ans:**

Yes. **(2 pts)**
Let $f(x) = cos(2x)$. Then

$$\begin{aligned}
\max_{0\le x\le 1} |\cos(2x) - P_n(x)| &= \max_{\substack{0\le x\le 1\\ 0\le \xi(x)\le 1}} \left| \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\ldots((x-x_n) \right| \ \textbf{(6 pts)}\\
&\le \frac{2^{n+1}}{(n+1)!} n! \frac{1}{n^{n+1}} \ \textbf{(6 pts)}\\
&= \frac{1}{n+1}\left(\frac{2}{n}\right)^{n+1}\\
&\le \frac{1}{n+1} \to 0 \ \text{ as } n \to \infty \textbf{(6 pts)}.
\end{aligned}$$

2. Denote by $P_{0,1,\cdots,k}(x)$ the Lagrange interpolating polynomial on the data set $(x_0, f(x_0)), (x_1, f(x_1), \cdots (x_k, f(x_k))$. Express $P_{0,1,\cdots,k}$ in terms of $P_{0,1,\cdots,j-1,j+1,\cdots,k}$ and $P_{0,1,\cdots,i-1,i+1,\cdots,k}$. Then verify your answer indeed is the Lagrange interpolating polynomial.

**Ans:**

Express

$$P_{0,1,\cdots,k}(x) = \frac{(x-x_j)P_{0,1,\cdots,j-1,j+1,\cdots,k}(x) - (x-x_i)P_{0,1,\cdots,i-1,i+1,\cdots,k}(x)}{x_i - x_j}. \ \textbf{(7 pts)}$$

For $t \ne i$ and $t \ne j$,

$$P_{0,1,\cdots,k}(x_t) = \frac{(x_t-x_j)f(x_t) - (x_t-x_i)f(x_t)}{x_i-x_j} = f(x_t). \ \textbf{(7 pts)}$$

For $t = i$,

$$P_{0,1,\cdots,k}(x_i) = \frac{(x_i-x_j)f(x_i)}{x_i-x_j} = f(x_i). \ \textbf{(2 pts)}$$

For $t = j$,

$$P_{0,1,\cdots,k}(x_j) = \frac{-(x_j-x_i)f(x_j)}{x_i-x_j} = f(x_j). \ \textbf{(2 pts)}$$

Finally, $\deg P_{0,1,\cdots,j-1,j+1,\cdots,k} \le k-1$ and $\deg P_{0,1,\cdots,i-1,i+1,\cdots,k} \le k-1$ imply that

$$\deg P_{0,1,\cdots,k} \le k. \ \textbf{(2 pts)}$$

3. A natural cubic spline $S$ on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & 0 \le x \le 1 \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & 1 \le x \le 2 \end{cases}$$

Find $b, c, d$.

**Ans:** Compute $S_0'$, $S_1'$, $S_0''$, and $S_1''$. **(2 pts)**

$$\begin{aligned} S_0'(1) &= S_1'(1) \textbf{ (3 pts)} \Rightarrow b = -1 \textbf{ (3 pts)} \\ S_0''(1) &= S_1''(1) \textbf{ (3 pts)} \Rightarrow c = -3 \textbf{ (3 pts)} \\ S_1''(2) &= 0 \textbf{ (3 pts)} \Rightarrow d = 1 \textbf{ (3 pts)} \end{aligned}$$

4. Suppose that we are to construct a piecewise polynomial interpolation $S(x)$ on the data $(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$, with additional continuity conditions for $S'$, $S''$ and $S'''$ on the interior nodes $x_1, \cdots, x_{n-1}$. If we use polynomials of the same degree on each of the interval $[x_0, x_1], \cdots, [x_{n-1}, x_n]$, what is the minimal degree needed in each interval? How many additional end conditions are needed? Count carefully and explain.

**Ans:**

There are $5n$ unknowns **(5 pts)** and $5n-3$ conditions **(5 pts)**. Therefore, the minimal degree is 4 **(5 pts)** and the number of additional boundary conditions is 3 **(5 pts)**.

5. Given four data $(x_i, \exp(-2x_i))$: $(0.3, 0.5488)$, $(0.4, 0.4493)$, $(0.5, 0.3679)$ and $(0.6, 0.3012)$ (you should generate the data yourself to avoid typo in inputting data). Use Inverse Interpolation to find the root of $x = \exp(-2x)$. You can use any algorithm for Lagrange interpolation. Extra credits for Neville's method (or divided difference, if anyone knows it). After finding $x$, check yourself that $x = \exp(-2x)$ in indeed satisfied in case of a bug in your code. Need not show the last part.

Hand in code, put all data within the code so that it can be executed immediately.

**Ans:**

$x = 0.426...$

Lagrange interpolation **(20 pts)**

Neville's method or divided difference **(extra 5 pts)**

C lagrange **(extra 5 pts)**

Name your codes in the same format as $s104000001.m$ or $s103000002.c$ and make sure it is executable/compilable.