Midterm 02

Dec 01, 2017.

1. (16 pts) Let P_n be the degree n interpolating polynomial of e^x on the uniformly spaced nodes $x_0, \cdots, x_n$ on $[0, 1]$ with $x_j = jh$, $h = 1/n$. Is it true that

$$\max_{0\le x\le 1} |e^x - P_n(x)| \le Ch^n$$

for some constant C independent of n? Explain.

Ans: Yes.

$$\begin{aligned}\max_{0\le x\le 1} |e^x - P_n(x)| &= \max_{\substack{0\le x\le 1\\ 0\le \xi(x)\le 1}} \left|\frac{e^{\xi(x)}}{(n+1)!}(x-x_0)\dots(x-x_n)\right| \textbf{ (8 pts)}\\ &\le \frac{e}{(n+1)!} n! h^{n+1} \textbf{ (6 pts)}\\ &\le eh^n. \textbf{ (2 pts)}\end{aligned}$$

2. (16 pts) Given four data $(0.6, \cos(0.6))$, $(0.7, \cos(0.7))$, $(0.8, \cos(0.8))$ and $(0.9, \cos(0.9))$, use Inverse Interpolation to find the root of $x = \cos(x)$. You can use any algorithm for Lagrange interpolation. Extra credits for Neville's method using loops.

Hand in your code, put all commands and data within the code so that it can be executed directly without further input.

Ans: The root is $x^* = 0.7390....$
Original Lagrange interpolation. **(16 pts)**
Neville's method. **(extra 3 pts)**
C language. **(extra 3 pts)**

3. (10 pts) Suppose that we are to construct a piecewise polynomial interpolation $S(x)$ on the data $(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$, with additional continuity conditions for S', S'', $S^{(3)}$ and $S^{(4)}$ on the interior nodes $x_1, \cdots, x_{n-1}$. If we use polynomials of the same degree on each of the interval $[x_0, x_1], \cdots, [x_{n-1}, x_n]$, what is the minimal degree needed in each interval? How many additional end conditions are needed? Explain.

Ans: There are $6n$ unknowns and $6n - 4$ conditions **(6 pts)**. Therefore, the minimal degree is 5 **(2 pts)** and the number of additional boundary conditions is 4 **(2 pts)**.

4. (10 pts) Use minimal number of data points among $f(x)$, $f(x+h)$, $f(x+2h)$, $f(x+3h), \cdots$ to approximate $f'(x)$ with $f'(x) - f_h'(x) = O(h^2)$. Give details.

Ans: Refer to the textbook for details. **(6 pts)**
$f_h'(x) = \frac{1}{2h}[-3f(x) + 4f(x+h) - f(x+2h)]$. **(4 pts)**

5. (16 pts) Find the constants a, b, c so that, the quadrature $\int_{-h}^{h} f(x)dx \approx h \cdot (af(-h) + bf(0) + cf(h))$ has highest degree of precision, where f is sufficiently smooth. Under the assumption (need not prove this assumption) that the error of this quadrature is of the form

$$\int_{-h}^{h} f(x)dx - h \cdot (af(-h) + bf(0) + cf(h)) = Kf^{(n)}(\xi)h^p,$$

find K, n and p.

Ans:
Details for finding a, b, c. **(2 pts)**
$a = \frac{1}{3}$ **(2 pts)**, $b = \frac{4}{3}$ **(2 pts)**, $c = \frac{1}{3}$ **(2 pts)**, DOP$= 3$.
Details for finding K, n, p. **(2 pts)**
$K = -\frac{1}{90}$ **(2 pts)**, $n = 4$ **(2 pts)**, $p = 5$ **(2 pts)**.

6. (16 pts + extra 8 pts max.) Use no more than 65 data points (such as $f(x_0), \cdots, f(x_{64})$) of your choice to evaluate $\int_0^1 f(x)dx$ numerically, where $f(x) = e^{-x}$. Two points for each correct digit.

Describe your method, write down your numerical answer and hand in your code.

Ans:
Composite Trapezoidal: 0.632133419301921 **(8 pts + extra 2 pts for C)**
Composite Simpson's : 0.632120559037870 **(16 pts + extra 3 pts for C)**
Composite Midpoint : 0.632114324941032 **(8 pts + extra 2 pts for C)**
Exact value : 0.632120558828558
A few possible alternatives: **((2 pts) $\times$ (correct digits) + extra partial credits for C)**
Composite Gaussian with $n = 1$: h^4
Composite Gaussian with $n = 2$: h^6
Simpson's + Richardson interpolation: h^6

7. (16 pts) Let $w(x) = e^{-x^2}$. Find a quadrature rule for $\int_{-1}^{1} w(x)f(x)dx$ that is exact for $f(x) = 1, x, x^2, x^3$ with minimal number of quadrature points. You can express your answer in terms of $a_0 = \int_{-1}^{1} e^{-x^2}dx$ and $a_2 = \int_{-1}^{1} x^2e^{-x^2}dx$ without explicitly evaluating a_0 and a_2 (which could be done).

You can instead do $w(x) = 1$ for partial credits.

Hint for both $w(x)$'s: Explore possible even/odd symmetry of c_i and x_i.

Ans:
Process for $w(x) = e^{-x^2}$. **(8 pts)**

$$\int_{-1}^{1} w(x)f(x)\,dx \approx \frac{a_0}{2}f(-\sqrt{\frac{a_2}{a_0}}) + \frac{a_0}{2}f(\sqrt{\frac{a_2}{a_0}}). \textbf{ (8 pts)}$$

Process for $w(x) = 1$. **(4 pts)**

$$\int_{-1}^{1} w(x) f(x)\, dx \approx f(-\sqrt{\frac{1}{3}}) + f(\sqrt{\frac{1}{3}}). \textbf{ (4 pts)}$$

8. (5 pts) Solve both roots for $x^2 - 1900 + 1 = 0$ to 14 correct digits. Explain how you find your answer (direct evaluation using 'calculator' will receive no credits).

   Ans:

$$x_1 = (1900 + \sqrt{1900^2 - 4})/2 \approx 1899.99947368406$$
$$x_2 = 1/x_1 \approx 5.26315935267612e - 04 \textbf{ (5 pts)}$$

9. (7 pts) Give a cubically convergent method to solve for $e^x - 1 = 0$. Give the formula and prove that it is cubically convergent (locally). If you cannot do it, do the same for a locally quadratically convergent method for partial credit.

   Ans:

   [Cubic]

   One solution is given by (there may be others)

$$x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)} \left[\frac{f(x)}{f'(x)}\right]^2. \quad \textbf{(2 pts)}$$

   Check that $g'(p) = g''(p) = 0$ and $g^{(3)}(p) \neq 0$, and then compute

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} = \frac{|g^{(3)}(p)|}{3!}. \quad \textbf{(5 pts)}$$

   [Quadratic]

$$x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)}. \quad \textbf{(1 pt)}$$

   Check that $g'(p) = 0$ and $g''(p) \neq 0$, and then compute

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(p)|}{2!}. \quad \textbf{(2 pts)}$$

10. (8 pts) Use any method to solve the nonlinear system of equations

$$\begin{aligned} 1x_1 + 2x_2 + 0.03 * \sin(x_1 + x_2) &= 4 \\ 5x_1 + 6x_2 + 0.07 * \cos(x_1 - x_2) &= 8 \end{aligned}$$

   Write your answer in the format of 'format long e' and hand in your code.

Let

$$g_1(\mathbf{x}) = -2x_2 - 0.03\sin(x_1+x_2)+4$$
$$g_2(\mathbf{x}) = -\frac{5}{6}x_1 - \frac{0.07}{6}\cos(x_1-x_2)+\frac{8}{6}$$

and

$$\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}),\ g_2(\mathbf{x}))^t$$
$$\bar{\mathbf{G}}(\mathbf{x}) = \alpha\mathbf{x} + (I-\alpha)\mathbf{G}(\mathbf{x})$$

where α is a 2×2 matrix and I is the identity matrix.

Consider

$$\mathbf{x}_{n+1} = \bar{\mathbf{G}}(\mathbf{x}_n) = \alpha\mathbf{x}_n + (I-\alpha)\mathbf{G}(\mathbf{x}_n)$$
$$\mathbf{x}_* = \bar{\mathbf{G}}(\mathbf{x}_*) = \alpha\mathbf{x}_* + (I-\alpha)\mathbf{G}(\mathbf{x}_*)$$
$$\overset{MVT}{\Rightarrow}\ \mathbf{x}_{n+1}-\mathbf{x}_* = \alpha(\mathbf{x}_n-\mathbf{x}_*) + (I-\alpha)\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{z_{n,1}}) & \frac{\partial g_1}{\partial x_2}(\mathbf{z_{n,1}}) \\ \frac{\partial g_2}{\partial x_1}(\mathbf{z_{n,2}}) & \frac{\partial g_2}{\partial x_2}(\mathbf{z_{n,2}}) \end{bmatrix}(\mathbf{x}_n-\mathbf{x}_*). --(1)$$

for some $\mathbf{z}_{n,1}$, $\mathbf{z}_{n,2}$ on the straight line path connecting $\mathbf{x}_n$ and $\mathbf{x}_*$. Since the nonlinear term is small, so the solution is near that of the linear part.

$$1x_1 + 2x_2 = 4$$
$$5x_1 + 6x_2 = 8$$

$$\Rightarrow x_1 = -2,\ x_2 = 3.$$

Therefore, we choose initial point $\mathbf{x}_0 = (-2,3)^t$ and approximate

$$\mathbf{z}_{n,1} \approx \mathbf{z}_{n,2} \approx \mathbf{x}_0. --(2)$$

Combine (1) and (2), then

$$\mathbf{x}_{n+1}-\mathbf{x}_* \approx \alpha(\mathbf{x}_n-\mathbf{x}_*) + (I-\alpha)D\mathbf{G}(\mathbf{x}_0)(\mathbf{x}_n-\mathbf{x}_*) = (\alpha + (I-\alpha)D\mathbf{G}(\mathbf{x}_0))(\mathbf{x}_n-\mathbf{x}_*).$$

To accelerate the convergence, we choose α such that

$$\alpha + (I-\alpha)D\mathbf{G}(\mathbf{x}_0) = 0$$
$$\Rightarrow \alpha = (D\mathbf{G}(\mathbf{x}_0)-I)^{-1}D\mathbf{G}(\mathbf{x}_0).$$

The approximation solution is $(-1.97005844315465e+00,\ 2.97238830860496e+00)^t$.

Another possible iteration: $x_{k+1} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 4-0.03... \\ 8-0.07... \end{bmatrix}.$

Code+Answer. **(8 pts, either all pts or no pts, partial credits only for typo in inputting data, outputting answer, or something else.)**
C language. **(extra 2 pts)**