Final Exam

Jan 12, 2018.

1. (15 pts) Use any desingularization method to evaluate the improper integral

$$\int_0^\infty \frac{1}{1+x^3}\,dx$$

with a composite quadrature rule of 4th order accuracy. Demonstrate numerically that the result is indeed 4th order accurate.

Answer:
Method 1.
Divide the integral into $\int_0^1$ and $\int_1^\infty$. Replace $x = s^{-q}$ in the second integral to obtain

$$\int_1^\infty \frac{1}{1+x^3}\,dx = q\int_0^1 \frac{s^{2q-1}}{s^{3q}+1}\,ds.$$

To make 4th order accuracy, we take $q = 1$. **(5 pts)**

Apply Simpson's rule or Gaussian quadrature, then we can demonstrate numerically the result

$$\log_2 \frac{I(100)-I(200)}{I(200)-I(400)} \approx 4. \textbf{ (10 pts)}$$

Method 2.
Use change of variable $t = \frac{1}{1+x}$.

C lagrange. **(Extra 3 pts)**

2. (15 pts) Consider discretizing

$$\begin{aligned}&(\partial_x^2+\partial_y^2)u(x,y) = f(x,y), \qquad (x,y)\in(0,5)\times(0,1)\\ &u = 0, \quad \text{on the boundary of } (0,5)\times(0,1)\end{aligned}$$

with uniformly spaced grids $0 = x_0 < x_1 < \cdots < x_{5N} = 5$, $0 = y_0 < y_1 < \cdots < y_N = 1$, $x_i - x_{i-1} = y_j - y_{j-1} = h = \frac{1}{N}$, using second order centered finite difference method. The following two possible ordering of the unknowns $u_{i,j}$,

$$(i,j) = (1,1)\to(1,2)\to\cdots\to(1,N-1)\to(2,1)\to(2,2)\to\cdots, \tag{1}$$

and

$$(i,j) = (1,1)\to(2,1)\to\cdots\to(5N-1,1)\to(1,2)\to(2,2)\to\cdots, \tag{2}$$

result in two linear systems $A^{(1)}u^{(1)} = f^{(1)}$ and $A^{(2)}u^{(2)} = f^{(2)}$. For large N, which linear system is faster to solve using Gaussian elimination without pivoting, combined with backward substitution? or are they the same? Explain.

Answer:
For the 1st ordering, the leading order term of number of operations is $5N^4$. **(7 pts)**
For the 2nd ordering, the leading order term of number of operations is $125N^4$. **(7 pts)**
Thus, the 1st one is faster to solve the system by Gaussian elimination. **(1 pt)**

3. (7+7 pts)

   (a) Perform the required row interchanges for the following linear system using scaled partial pivoting:

$$\begin{aligned} x_2 + \ x_3 &= 4 \\ x_1 - \ 2x_2 - \ x_3 &= 5 \\ x_1 - \ x_2 + \ x_3 &= 6 \end{aligned} , \tag{3}$$

   Give all details and the final linear system. Need not solve it.

   Answer:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 1 & -2 & -1 & 5 \\ 1 & -1 & 1 & 6 \end{array}\right] \xrightarrow{E_1 \leftrightarrow E_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 1 & -2 & -1 & 5 \\ 0 & 1 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & 1 & 4 \end{array}\right] \ \textbf{(4 pts)}$$

$$\xrightarrow{E_2 \leftrightarrow E_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & -2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & 3 \end{array}\right] \ \textbf{(3 pts)}$$

   (b) Find P, L, U in the factorization $PA = LU$ for the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \end{pmatrix} \tag{4}$$

   where P is the permutation matrix corresponding to partial pivoting.

   Answer:
   Finding U:

$$U = \begin{pmatrix} -1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \ \textbf{(3 pts)}$$

   Finding P:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ \textbf{(2 pts)}$$

Finding L:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \textbf{(2 pts)}$$

4. (15 pts) True or False? Explain.

   Suppose that $A = M - N$ is nonsingular. Then the iteration $Mx^{(k+1)} = b + Nx^{(k)}$ converges for any $x^{(0)}$ if M is nonsingular and $T = M^{-1}N$ satisfies $\|T\| < 1$ for some matrix norm $\|\cdot\|$.

   Answer:
   True. Let x be the exact solution and $E_k := x^{(k)} - x$. Then

$$\|E_k\| = \|x^{(k)} - x\| = \|(Tx^{(k-1)} + M^{-1}b) - (Tx + M^{-1}b)\| = \|Tx^{(k-1)} - Tx\| = \|TE_{k-1}\|$$

$$\leq \|T\|\|E_{k-1}\| \leq \ldots \leq \|T\|^k\|E_0\| \overset{\|T\|<1}{\to} 0$$

   as $k \to \infty$. **(15 pts)**

5. (15 pts) Write down the definition of 'condition number'. Derive a related inequality that gives an estimate of relative error of the solution of a linear system.

   Answer:
   See Definition 7.28. for the definition. **(3 pts)**
   See Theorem 7.27 for the inequality.
   Correct inequality. **(3 pts)**
   Details of derivations:$\|b\| \leq \|A\|\|x\|$, $\|\Delta x\| \leq \|A^{-1}\|\|\Delta b\|$. **(9 pts)**

6. (15 pts) Consider the integral equation

$$u(x_i) = f(x_i) + \int_0^1 K(x_i, t)\, u(t)\, dt, \quad i = 0, \cdots, M, \tag{5}$$

   where $0 = x_0 < x_1 < \cdots < x_M = 1$, with $x_j - x_{j-1} = h = \frac{1}{M}$, $f(x) = x^2$, $K(x,t) = -\frac{1}{3}e^{-|x-t|}$ and $\int_0^1\, dt$ is evaluated using trapezoidal rule.

   Form the linear system, find an convergent iterative method, explain why your method converges. Need not solve it.

   Answer:
   The Jacobi iteration will converge. Consider

$$\begin{aligned} \|T_j\|_\infty = \max_{i=1,\ldots M+1} \sum_{j=1}^{M+1} |(T_j)_{ij}| \quad &\approx \quad \max_{i=1,\ldots M+1} \frac{\frac{1}{3}}{\frac{h}{3}+1} \int_0^1 e^{-|x_i-t|}\, dt \\ &= \quad \max_{i=1,\ldots M+1} \frac{1}{h+3}(2 - e^{-x_i} - e^{x_i-1}) \\ &\approx \quad \max_{i=1,\ldots M+1} \frac{1}{3}(2 - e^{-x_i} - e^{x_i-1}) \end{aligned}$$

$$
\begin{aligned}
&\approx \max_{x\in[0,1]} \frac{1}{3}(2 - e^{-x} - e^{x-1}) \\
&= \frac{2}{3}\left(1 - \frac{1}{\sqrt{e}}\right) < 1.
\end{aligned}
$$

Find the linear system. **(5 pts)**
Find an convergent iterative method. **(5 pts)**
Explain why your method converges. **(5 pts)**

7. (15 pts) Find the polynomial $p(x)$ that minimizes $\int_0^1 (\sin x - p(x))^2 dx$ among all odd polynomials ($p(-x) = -p(x)$) with degree less than or equal to 5. Derive the linear system and need not solve it numerically.

   Answer:
   Let $p(x) = a_1 x + a_3 x^3 + a_5 x^5$. Then the minimization problem results in the linear system

$$
\begin{aligned}
\frac{1}{3}a_1 + \frac{1}{5}a_3 + \frac{1}{7}a_5 &= \int_0^1 x \sin x \, dx \ \textbf{(5 pts)} \\
\frac{1}{5}a_1 + \frac{1}{7}a_3 + \frac{1}{9}a_5 &= \int_0^1 x^3 \sin x \, dx \ \textbf{(5 pts)} \\
\frac{1}{7}a_1 + \frac{1}{9}a_3 + \frac{1}{11}a_5 &= \int_0^1 x^5 \sin x \, dx. \ \textbf{(5 pts)}
\end{aligned}
$$

8. (8 pts) Find the constants a, b, c so that, the quadrature $\displaystyle\int_{-h}^{h} f(x)dx \approx h \cdot (af(-h) + bf(0) + cf(h))$ has highest degree of precision, where f is sufficiently smooth. Under the assumption (need not prove this assumption) that the error of this quadrature is of the form

$$
\int_{-h}^{h} f(x)dx - h \cdot (af(-h) + bf(0) + cf(h)) = Kf^{(n)}(\xi)h^p,
$$

   find K, n and p.

   Ans:
   Details for finding a, b, c. **(1 pt)**
   $a = \frac{1}{3}$ **(1 pt)**, $b = \frac{4}{3}$ **(1 pt)**, $c = \frac{1}{3}$ **(1 pt)**, DOP= 3.
   Details for finding K, n, p. **(1 pt)**
   $K = -\frac{1}{90}$ **(1 pt)**, $n = 4$ **(1 pt)**, $p = 5$ **(1 pt)**.

9. (8 pts) Use any method to solve the nonlinear system of equations

$$
\begin{aligned}
1x_1 + 2x_2 + 0.03 * \sin(x_1 + x_2) &= 4 \\
5x_1 + 6x_2 + 0.07 * \cos(x_1 - x_2) &= 8
\end{aligned}
$$

   Write your answer in the format of 'format long e' and hand in your code.

Answer:
Let

$$g_1(\mathbf{x}) = -2x_2 - 0.03\sin(x_1 + x_2) + 4$$
$$g_2(\mathbf{x}) = -\frac{5}{6}x_1 - \frac{0.07}{6}\cos(x_1 - x_2) + \frac{8}{6}$$

and

$$\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}),\ g_2(\mathbf{x}))^t$$
$$\bar{\mathbf{G}}(\mathbf{x}) = \alpha\mathbf{x} + (I - \alpha)\mathbf{G}(\mathbf{x})$$

where α is a 2×2 matrix and I is the identity matrix.
Consider

$$\mathbf{x}_{n+1} = \bar{\mathbf{G}}(\mathbf{x}_n) = \alpha\mathbf{x}_n + (I - \alpha)\mathbf{G}(\mathbf{x}_n)$$
$$\mathbf{x}_* = \bar{\mathbf{G}}(\mathbf{x}_*) = \alpha\mathbf{x}_* + (I - \alpha)\mathbf{G}(\mathbf{x}_*)$$
$$\overset{MVT}{\Rightarrow} \mathbf{x}_{n+1} - \mathbf{x}_* = \alpha(\mathbf{x}_n - \mathbf{x}_*) + (I - \alpha)\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{z_{n,1}}) & \frac{\partial g_1}{\partial x_2}(\mathbf{z_{n,1}}) \\ \frac{\partial g_2}{\partial x_1}(\mathbf{z_{n,2}}) & \frac{\partial g_2}{\partial x_2}(\mathbf{z_{n,2}}) \end{bmatrix}(\mathbf{x}_n - \mathbf{x}_*). --(1)$$

for some $\mathbf{z}_{n,1}$, $\mathbf{z}_{n,2}$ on the straight line path connecting $\mathbf{x}_n$ and $\mathbf{x}_*$. Since the nonlinear term is small, so the solution is near that of the linear part.

$$1x_1 + 2x_2 = 4$$
$$5x_1 + 6x_2 = 8$$
$$\Rightarrow x_1 = -2,\ x_2 = 3.$$

Therefore, we choose initial point $\mathbf{x}_0 = (-2, 3)^t$ and approximate

$$\mathbf{z}_{n,1} \approx \mathbf{z}_{n,2} \approx \mathbf{x}_0. --(2)$$

Combine (1) and (2), then

$$\mathbf{x}_{n+1} - \mathbf{x}_* \approx \alpha(\mathbf{x}_n - \mathbf{x}_*) + (I - \alpha)D\mathbf{G}(\mathbf{x}_0)(\mathbf{x}_n - \mathbf{x}_*) = (\alpha + (I - \alpha)D\mathbf{G}(\mathbf{x}_0))(\mathbf{x}_n - \mathbf{x}_*).$$

To accelerate the convergence, we choose α such that

$$\alpha + (I - \alpha)D\mathbf{G}(\mathbf{x}_0) = 0$$
$$\Rightarrow \alpha = (D\mathbf{G}(\mathbf{x}_0) - I)^{-1}D\mathbf{G}(\mathbf{x}_0).$$

The approximation solution is $(-1.97005844315465e+00,\ 2.97238830860496e+00)^t$.

Another possible iteration: $x_{k+1} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 4 - 0.03... \\ 8 - 0.07... \end{bmatrix}$.

Code+Answer. **(8 pts, either all pts or no pts, partial credits only for typo in inputting data, outputting answer, or something else.)**
C language. **(Extra 2 pts)**