HW16

HW #1. §7.4 #7c.

The matrix are in part (c) is tridiagonal and positive-definite. (Why?)

For $\omega = 1.1$, we have

$$\mathbf{x}^{(2)} = (-0.71885,\ 2.81882,\ -0.28097,\ -2.23542)^t,$$

$$\|\mathbf{x}^{(2)} - \mathbf{x}\|_\infty = 0.078797.$$

$$\mathbf{x}^{(10)} = (-0.79765,\ 2.79529,\ -0.25882,\ -2.25176)^t,$$

$$\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 7.0320e - 07.$$

For the optimal $\omega = 1.153499$ (Thm.7.26), we have

$$\mathbf{x}^{(2)} = (-0.84712,\ 2.92031,\ -0.24927,\ -2.20025)^t,$$

$$\|\mathbf{x}^{(2)} - \mathbf{x}\|_\infty = 0.12502.$$

$$\mathbf{x}^{(10)} = (-0.79765,\ 2.79529,\ -0.25882,\ -2.25176)^t,$$

$$\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 4.1306e - 07 < 7.0320e - 07.$$

HW #1. §7.4 #13.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T_ω. Then

$$\begin{aligned}\prod_{i=1}^{n} \lambda_i &= \det T_\omega = \det\left((D-\omega L)^{-1}[(1-\omega)D+\omega U]\right) \\ &= \det(D-\omega L)^{-1}\det((1-\omega)D+\omega U) = \det(D^{-1})\det((1-\omega)D) \\ &= \left(\frac{1}{(a_{11}a_{22}\ldots a_{nn})}\right)\left((1-\omega)^n a_{11}a_{22}\ldots a_{nn}\right) = (1-\omega)^n.\end{aligned}$$

Thus

$$\rho(T_\omega) = \max_{1\le i\le n} |\lambda_i| \ge |\omega - 1|.$$

HW #3.

Consider

$$
\begin{aligned}
\|T_j\|_\infty = \max_{i=1,\dots m+1} \sum_{j=1}^{m+1} |(T_j)_{ij}| &\approx \max_{i=1,\dots m+1} \frac{\frac{1}{2}}{\frac{h}{2}+1} \int_0^1 e^{-|x_i - t|}\, dt \\
&= \max_{i=1,\dots m+1} \frac{1}{h+2}(2 - e^{-x_i} - e^{x_i - 1}) \\
&\approx \max_{i=1,\dots m+1} \frac{1}{2}(2 - e^{-x_i} - e^{x_i - 1}) \\
&\approx \max_{x\in[0,1]} \frac{1}{2}(2 - e^{-x} - e^{x-1}) \\
&= 1 - \frac{1}{\sqrt{e}}.
\end{aligned}
$$

Note that

$$
\|e^{(k)}\| \le \|T^k\| \|e^{(0)}\| \approx \left(1 - \frac{1}{\sqrt{e}}\right)^k \le h^2
$$

$$
\Rightarrow k \ge \frac{2\log m}{\log \frac{\sqrt{e}}{\sqrt{e}-1}} = O(\log m).
$$

And, the leading order of operation count for each Jacobi iteration

$$
x_i^{(k)} = (b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k-1)})/a_{ii}
$$

is $O(m^2)$.

Therefore, the leading order of total operation count is $O(m^2 \log m)$, which is smaller than $O(m^3)$ of GE/LU.

HW #4.

All the details are left as exercises. The leading order of operation count is called LD for short.

- (a)
  - Jacobi: $x_i^{(k)} = (b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k-1)})/a_{ii} \Rightarrow LD = 5N^2$
  - Gauss-Seidel: $x_i^{(k)} = (b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)})/a_{ii} \Rightarrow LD = 5N^2$
  - SOR: $x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \omega(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)})/a_{ii} \Rightarrow LD = 7N^2$

    Or, $x_i^{(k)} = x_i^{(k-1)} + \omega[(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)})/a_{ii} - x_i^{(k-1)}] \Rightarrow LD = 6N^2$

- (b)
  - Step 1. Claim that if $\rho(T) < 1$, then $\|T^k\| \le c\left(\frac{\rho(T)+1}{2}\right)^k$ for some constant c. (Proved in class.)
  - Step 2. Since $\rho(T) = 1 - rh^2$ for some $r > 0$ (the given fact), we have $\|T^k\| \le c(1-sh^2)^k$ for some $s > 0$. Note that

    $$\|e^{(k)}\| \le \|T^k\|\|e^{(0)}\| \le c(1-sh^2)^k.$$

    Consider

    $$\begin{aligned} & c(1-sh^2)^k \le h^2 \\ \Rightarrow\ & k \ge \frac{2\log h - \log c}{\log(1-sh^2)} \approx \frac{2\log h - \log c}{-sh^2} = \frac{2\log h^{-1} + \log c}{sh^2} = \frac{(2\log N + \log c)N^2}{s} = O(N^2\log N). \end{aligned}$$

    Therefore, $LD = 5N^2 \cdot O(N^2\log N) = O(N^4\log N)$ for both Jacobi and Gauss-Siedel, which is close to $O(N^4)$ of GE/LU.

- (c) Consider

  $$\begin{aligned} & c(1-sh)^k \le h^2 \\ \Rightarrow\ & k \ge \frac{2\log h - \log c}{\log(1-sh)} \approx \frac{2\log h - \log c}{-sh} = \frac{2\log h^{-1} + \log c}{sh} = \frac{(2\log N + \log c)N}{s} = O(N\log N). \end{aligned}$$

  Therefore, $LD = O(N^2)\cdot O(N\log N) = O(N^3\log N)$ for such SOR, which is smaller than $O(N^4)$ of GE/LU.

- (d)
  - (a)
    - Jacobi: $7N^3$
    - Gauss-Seidel: $7N^3$
    - SOR: $9N^3$ or $8N^3$
  - (b) $k = O(N^2\log N) \Rightarrow LD = O(N^5\log N)$, which is smaller than $O(N^7)$ of GE/LU.
  - (c) $k = O(N\log N) \Rightarrow LD = O(N^4\log N)$, which is smaller than $O(N^7)$ of GE/LU.

HW #5.

The Jacobi version of SOR:

$$
\begin{aligned}
&Dx^{(k)} = [(1-\omega)D + \omega(L+U)]x^{(k-1)} + \omega b \\
\Rightarrow\ &T_{\omega,j} = D^{-1}[(1-\omega)D + \omega(L+U)] = (1-\omega)I + \omega D^{-1}(L+U) = (1-\omega)I + \omega T_j.
\end{aligned}
$$

HW #6. §7.5 #1ac. #3ac.

- #1a. 50
- #1c. $600, 002$
- #3a. 8.571429×10^{-4}, 1.238095×10^{-2}
- #3c. 0.04, 0.08

HW #6. §7.5 #11. #12a.

- #11.

$$
\begin{aligned}
& \exists\, x \text{ s.t. } \|x\| = 1,\ Bx = 0 \\
\Rightarrow\ & 1 = \|x\| = \|A^{-1}Ax\| \le \|A^{-1}\|\|Ax\| \\
\Rightarrow\ & \frac{1}{\|A^{-1}\|} \le \|Ax\| = \|(A-B)x\| \le \|A-B\| \\
\Rightarrow\ & \frac{1}{K(A)} \le \frac{\|A-B\|}{\|A\|}
\end{aligned}
$$

- #12a. With $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, we have $K_\infty(A) \ge 30001.$ (Exercise!)

HW #7.

Run this code

```
format long
for n=6:12
    cond(hilb(n), inf)
end
```

and the results will be

- $K_\infty(H^{(6)}) = 29070279.0011431$
- $K_\infty(H^{(7)}) = 985194889.826450$
- $K_\infty(H^{(8)}) = 33872791940.3055$
- $K_\infty(H^{(9)}) = 1099651019168.29$
- $K_\infty(H^{(10)}) = 35351252337165.5$
- $K_\infty(H^{(11)}) = 1.22715686249259e + 15$
- $K_\infty(H^{(12)}) = 37092156700476872, \text{ matrix singular to machine precision}$

HW #8. §8.1 #2.

The least-squares polynomial of degree two is $P_2(x) = 0.4066667 + 1.154848x + 0.03484848x^2$, with $E = 1.7035$.

HW #8. §8.1 #14.

For each $i = 1, \ldots, n+1$ and $j = 1, \ldots, n+1$, $a_{ij} = a_{ji} = \sum_{k=1}^{m} x_k^{i+j-2}$, so $A = (a_{ij})$ is symmetric.

Suppose A is singular and $\mathbf{c} \neq \mathbf{0}$ satisfies $\mathbf{c}^t A \mathbf{c} = \mathbf{0}$. Then

$$0 = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} c_i c_j = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \left(\sum_{k=1}^{m} x_k^{i+j-2} \right) c_i c_j = \sum_{k=1}^{m} \left[\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} c_i c_j x_k^{i+j-2} \right],$$

so

$$\sum_{k=1}^{m} \left(\sum_{i=1}^{n+1} c_i x_k^{i-1} \right)^2 = 0.$$

Define $P(x) = c_1 + c_2 x + \ldots + c_{n+1} x^n$. Then $\sum_{k=1}^{m} [P(x_k)]^2 = 0$ and $P(x)$ has roots $x_1, \ldots, x_m$. Since the roots are distinct and $m > n$, $P(x)$ must be the zero polynomial. Thus, $c_1 = c_2 = \ldots = c_{n+1} = 0$, and A must be nonsingular.

HW #9.

The details are left as exercises.

$$\sum_{k=0}^{n} \frac{1}{j+k+1} a_k = \int_0^1 f(x) x^j \, dx, \; j = 0, \dots, n.$$

Note that the coefficients $\frac{1}{j+k+1}$, $j, k = 0, \dots, n$, compose the Hilbert matrix, which is ill-conditioned.