HW7

Textbook §3.1 #6a.

```
format long
d = 3;
n = d;
z = 0.43;
x = [0.25 0.5 0.75 0];
f = [1.64872 2.71828 4.48169 1];

for k = 1:n+1
    L(k) = 1;
    for i = 1:n+1
        if ( i != k )
            L(k) =  L(k)*(z-x(i))/(x(k)-x(i));
        end
    end
end

P = 0;
for k = 1:n+1
    P = P + f(k)*L(k);
end
P
```

For $d = 1$, $P = 2.41880320000000$.

For $d = 2$, $P = 2.34886312000000$.

For $d = 3$, $P = 2.36060473408000$.

Textbook §3.1 #9.

Let $P_3(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ with $a_3 = 6$.

Solve

$$\begin{aligned} a_0 &= P_3(0) = 0 \\ \frac{a_3}{8} + \frac{a_2}{4} + \frac{a_1}{2} + a_0 &= P_3(0.5) = y \\ a_3 + a_2 + a_1 + a_0 &= P_3(1) = 3 \\ 8a_3 + 4a_2 + 2a_1 + a_0 &= P_3(2) = 2. \end{aligned}$$

Then we find $y = 4.25$ (Exercise!).

Textbook §3.1 #10.

Let $P_2(x) = a_2 x^2 + a_1 x + a_0$.

Solve

$$\begin{aligned} f(0) &= P_2(0) \\ f(1) &= P_2(1) \\ f(0.5) - P_2(0.5) &= -0.25. \end{aligned}$$

Then we find $a_0 = 0, a_1 = 3, a_2 = -3$ (Exercise!).

Finally solve $P_2(x_1) = f(x_1)$. Then we find $x_1 = \frac{3 \pm \sqrt{5}}{6}$ (Exercise!).

Therefore, the largest value of x_1 is

$$x_1 = \frac{3 + \sqrt{5}}{6} = 0.872677996...$$

Textbook §3.1 #13a.

Construct the Lagrange interpolating polynomial.

$$\begin{aligned}
L_{2,0}(x) &= \frac{(x-0.3)(x-0.6)}{(0-0.3)(0-0.6)} \\
L_{2,1}(x) &= \frac{(x-0)(x-0.6)}{(0.3-0)(0.3-0.6)} \\
L_{2,2}(x) &= \frac{(x-0)(x-0.3)}{(0.6-0)(0.6-0.3)} \\
P_2(x) &= \sum_{k=0}^{2} f(x_k)L_{2,k}(x) \\
&= \frac{f(0)-2f(0.3)+f(0.6)}{0.18}x^2 + \frac{-0.9f(0)+1.2f(0.3)-0.3f(0.6)}{0.18}x + \frac{0.18f(0)}{0.18} \\
&= -11.22017744\ldots x^2 + 3.80821060\ldots x + 1
\end{aligned}$$

Find a bound for the absolute error on $[x_0, x_1]$. By Theorem 3.3

$$\begin{aligned}
error &= \left|\frac{f'''(\xi(x))}{3!}(x-0)(x-0.3)(x-0.6)\right|, \quad 0 \le \xi(x) \le 0.6 \\
&\le \left|\frac{-e^{2\xi(x)}(46\cos 3\xi(x) + 9\sin 3\xi(x))}{6}x(x-0.3)(x-0.6)\right| \\
&\le \frac{1}{6}\max_{[0,0.6]} |-e^{2x}(46\cos 3x + 9\sin 3x)| \max_{[0,0.6]} |x(x-0.3)(x-0.6)| \\
&= \frac{1}{6}\left|f'''\left(\frac{1}{3}\arctan(\frac{119}{120})\right)\right| g(0.3-\sqrt{0.03}) \\
&= 0.113712937670298...
\end{aligned}$$

where $g(x) = |x(x-0.3)(x-0.6)|$ and thoes inputted points are critical points (Check it!).

Textbook §3.1 #17.

Let $f(x) = \log_{10} x$ and $x_j = 1 + jh, j = 0, 1, \ldots, 9$.

By Theorem 3.3, for $x_j \le x \le x_{j+1}$, we have

$$
\begin{aligned}
error &= \left| \frac{f''(\xi(x))}{2!}(x - x_j)(x - x_{j+1}) \right|, \quad x_j \le \xi(x) \le x_{j+1} \\
&= \left| \frac{\frac{-1}{(\ln 10)\xi(x)^2}}{2}(x - (1 + jh))(x - (1 + (j + 1)h)) \right| \\
&\le \frac{\frac{1}{\ln 10}}{2} \frac{h^2}{4} \text{ (Exercise!)} \\
&\le 10^{-6} \\
\Rightarrow h &\le \sqrt{8(\ln 10)10^{-6}} = 0.004291932\ldots
\end{aligned}
$$

Choose $h = 0.004$, then $n = 9/h = 2250$.

HW #3.

(a) For any $a \le x \le b$, $a + jh \le x \le a + (j+1)h$ for some $j = 0, 1, \dots, n-1$.

For $i = 0, 1, \dots, j$,

$$(j - i)h \le |x - x_i| \le (j + 1 - i)h.$$

For $i = j + 1, \dots, n$,

$$(i - j - 1)h \le |x - x_i| \le (i - j)h.$$

Therefore,

$$|(x - x_0) \dots (x - x_n)| \le \prod_{i=0}^{j} (j + 1 - i)h \prod_{i=j+1}^{n} (i - j)h \overset{(Why?)}{\le} n! h^{n+1}.$$

(b) By Theorem 3.3,

$$\begin{aligned} \max_{0 \le x \le 1} |e^x - P_n(x)| &= \max_{\substack{0 \le x \le 1 \\ 0 \le \xi(x) \le 1}} \left| \frac{e^{\xi(x)}}{(n+1)!} (x - x_0) \dots (x - x_n) \right| \\ &\overset{(a)}{\le} \frac{e}{(n+1)!} n! h^{n+1} \\ &\le e h^n \to 0 \end{aligned}$$

as $n \to \infty$.