Homework Assignment for Week 11

Assigned Nov 25, 2011.

1. Section 7.3: Problems 17(a,c), 18(a,c), 22.

2. Reading instruction for section 7.3:

   Read and understand meanings of Jacobi, Gauss-Siedel and SOR methods.

   SKIP the following: Theorem 7.22, Definition 7.23 through eq. (7.16) (read your class note for derivation of SOR instead), Theorem 7.24, 7.25, 7.26.

3. Let A be the matrix resulted from discretizing

$$\begin{aligned}&(\partial_x^2 + \partial_y^2)u(x,y) = f(x,y), \qquad (x,y) \in (0,1)^2 \\ &u = 0, \quad \text{on the boundary of } (0,1)^2\end{aligned} \tag{1}$$

   with uniformly spaced grids $0 = x_0 < x_1 < \cdots < x_N = 1$, $0 = y_0 < y_1 < \cdots < y_N = 1$, $x_i - x_{i-1} = y_j - y_{j-1} = h = 1/N$, using second order centered finite difference method. Study your class note and try to derive the matrix A by yourself.

   (a) Recall that LU decomposition for this A requires N^4 multiplications from previous homework problems. Estimate the number of multiplications (to leading order, also for rest of this problem) for solving $Lz = b$ and $Ux = z$, respectively.

   (b) Estimate numbers of multiplications needed for one iteration of Jacobi, Gauss Seidel and SOR on this A, respectively.

   (c) It is a fact (the proof is beyond this course) that **for this** A, $\rho(T_j) \approx 1 - \frac{\pi^2}{2}h^2$, $\rho(T_g) \approx 1 - \pi^2 h^2 \approx \rho(T_j)^2$. Note that both numbers are very close to 1. Take this fact for granted, estimate the number of iterations Jacobi and Gauss-Siedel take to reach $\|e^{(k)}\| = h^2$, respectively (assuming $\|e^{(0)}\| = 1$) where $e^{(k)} = u^{(k)} - u_e$. Then estimate the total number of multiplications needed for Jacobi and Gauss-Siedel, respectively. You will need that $\log(1 + x) \approx x$ for $|x| << 1$.

   (d) It is another fact (the proof is also beyond this course) that with properly chosen optimal $\omega = \omega^* \approx 2 - 2\pi h$, we will have $\rho(T_{\omega^*}) \approx 1 - 2\pi h$ for SOR (for this A). Take this fact for granted again, estimate the number of iterations and total multiplications the optimal SOR takes to reach $\|e^{(k)}\| = h^2$ with $\|e^{(0)}\| = 1$.

   (e) The facts that $\rho(T_j) \approx 1 - C_1 h^2$, $\rho(T_g) \approx 1 - C_2 h^2$ and $\rho(T_{\omega^*}) \approx 1 - C_3 h$ remain valid in the 3D version of (1), with different constants C_i. Repeat the above problems for the 3D case.

   The purpose of this problem is to show you that, beating Gauss Elimination/LU decomposition with iterative methods is possible, but sometimes non-trivial.

4. Derive the Jacobi version of SOR. Express $T_{\omega,j}$ in terms of T_j.

5. Continuation on problem 4 of Homework 10, which requires handing in your code, is postponed till after the midterm.