Homework Assignment for Week 02

1. Section 1.3: Problems 3, 4, 7, 14, 15.

2. Another express of π is given by the Wallis formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{(2n)^2}{(2n-1)(2n+1)} \right).$$

   There are two ways of computing the N-term approximation, namely

$$\frac{\pi_N}{2} = \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2N\cdot 2N}{(2N-1)\cdot(2N+1)}\right) \quad \text{or} \quad \frac{4^N (N!)^2}{\left(\prod_{n=1}^{N}(2n-1)\right)\cdot\left(\prod_{n=1}^{N}(2n+1)\right)}$$

   Which one is better in terms of the overflow issue? What is the rate of convergence?

   Hint: The rate of convergence can be analyzed analytically, which may be easier for you. But if time permits, you should also explore numerically by plotting $|\pi - \pi_N|$ as a function of N. For example, the two methods in problem 4 of section 1.3 give algebraic rate (N^{-k}) and exponential rate (q^{-N}), respectively. The difference can easily be observed by plotting the results. The former appears as a straight line in the 'loglog' plot on matlab and the latter appears as a straight line in 'semilogy' plot.

3. Consider the following recursive relation

$$x_1 = a, \quad x_2 = \frac{a}{2}, \quad x_n = \frac{5}{2}x_{n-1} - x_{n-2}$$

   (a) What is the exact solution?

   (b) With $a = 1/3$, is it stable? How fast does the relative error grow/decay in n as n gets larger? Analyze it and verify your answer numerically.

   (c) Do the same for $a = 1$ and explain your observation.

4. Derive a finite difference approximation for $f'(x)$ and $f''(x)$ respectively from $f(x-h)$, $f(x)$ and $f(x+2h)$. This is another demonstration why (symmetric) centered difference works better in terms of accuracy.