Homework Assignment for Week 09

Assigned Nov 13, 2009. Due Nov 20, 2009.

1. Section 4.7: Problems 1, 3(a), 4, 5, 8, 9.

   Hint for Problem 3(a): sin, P_1 and P_3 are odd functions, while P_0, P_2 and P_4 are even.

   For problem 8, also find the least square quadratic approximation of e^x on $[-1,1]$ with respect to the inner product induced by $w(x)$.

2. Denote by $H^{(n)}$ the $n \times n$ Hilbert matrix with entries $H^{(n)}_{i,j} = \frac{1}{i+j-1}$, $1 \le i,j \le n$. This matrix arises naturally from least square approximation over the interval $[0,1]$.

   (a) (Extra credit, due Nov 27) Show that $H^{(n)}$ is non-singular.

   Hint: The basis functions used in the least square approximation are linearly independent. Use this to show that, by changing this set of basis functions to another set of orthonormal basis functions, one can find a non-singular $n \times n$ matrix V such that $H^{(n)} = V^* V$.

   (b) Take an arbitrary vector $\boldsymbol{x}_T \in \mathbb{R}^n$. Perform the matrix-vector multiplication to get $\boldsymbol{b} = H^{(n)}\boldsymbol{x}_T$. Then solve for $\boldsymbol{x}$ from the equation $H^{(n)}\boldsymbol{x} = \boldsymbol{b}$, denote the numerical solution by $\boldsymbol{x}_A$, and compare $\boldsymbol{x}_A$ with $\boldsymbol{x}_T$ to find the relative error. Do this for $n = 5$, 10 and 15 respectively.

3. Challenge of the week with extra credit. Due Nov 27.

   The Legendre function P_n defined by (4.114) are also known as the eigenfunctions of the eigenvalue problem

$$\frac{d}{dx}\Big((x^2-1)\frac{d}{dx}P(x)\Big) = \lambda P(x), \qquad -1 \le x \le 1. \tag{1}$$

   (a) Verify that $P_n(x)$ satisfies (1) with corresponding eigenvalue $\lambda_n = n(n+1)$.

   Hint: Let $q(x) = (x^2-1)^n$, show that $(x^2-1)q'(x) = 2nxq(x)$. Keep differentiate $n+1$ times and use Leibniz rule.

   (b) Use (1) to verify that $\langle P_m, P_n \rangle = 0$ if $n \ne m$.

   Hint: why are eigenvectors of a symmetric matrix corresponding to different eigenvalues, orthogonal to each other?

   (c) Use (4.114) and the mean value theorem to show (4.119).