21. Suppose two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Two formulas are available to find the x-intercept of the line:

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} \quad \text{and} \quad x = x_0 - \frac{(x_1 - x_0)y_0}{y_1 - y_0}.$$

a. Show that both formulas are algebraically correct.

b. Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and three-digit rounding arithmetic to compute the x-intercept both ways. Which method is better, and why?

22. The Taylor polynomial of degree n for $f(x) = e^x$ is $\sum_{i=0}^{n}(x^i/i!)$. Use the Taylor polynomial of degree nine and three-digit chopping arithmetic to find an approximation to e^{-5} by each of the following methods.

a. $e^{-5} \approx \sum_{i=0}^{9} \frac{(-5)^i}{i!} = \sum_{i=0}^{9} \frac{(-1)^i 5^i}{i!}$

b. $e^{-5} = \frac{1}{e^5} \approx \frac{1}{\sum_{i=0}^{9} \frac{5^i}{i!}}.$

c. An approximate value of e^{-5} correct to three digits is 6.74×10^{-3}. Which formula, (a) or (b), gives the most accuracy, and why?

23. The two-by-two linear system

$$ax + by = e,$$
$$cx + dy = f,$$

where a, b, c, d, e, f are given, can be solved for x and y as follows:

$$\text{set } m = \frac{c}{a}, \quad \text{provided } a \neq 0;$$
$$d_1 = d - mb;$$
$$f_1 = f - me;$$
$$y = \frac{f_1}{d_1};$$
$$x = \frac{(e - by)}{a}.$$

Solve the following linear systems using four-digit rounding arithmetic.

a. $1.130x - 6.990y = 14.20$
$1.013x - 6.099y = 14.22$

b. $8.110x + 12.20y = -0.1370$
$-18.11x + 112.2y = -0.1376$

24. Repeat Exercise 23 using four-digit chopping arithmetic.

25. a. Show that the polynomial nesting technique described in Example 6 can also be applied to the evaluation of

$$f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^x - 1.99.$$

b. Use three-digit rounding arithmetic, the assumption that $e^{1.53} = 4.62$, and the fact that $e^{nx} = (e^x)^n$ to evaluate $f(1.53)$ as given in part (a).

12. The number e can be defined by $e = \sum_{n=0}^{\infty}(1/n!)$, where $n! = n(n-1)\cdots 2\cdot 1$ for $n \neq 0$ and $0! = 1$. Compute the absolute error and relative error in the following approximations of e:

  a. $\displaystyle\sum_{n=0}^{5}\frac{1}{n!}$ b. $\displaystyle\sum_{n=0}^{10}\frac{1}{n!}$

13. Let

$$f(x) = \frac{x\cos x - \sin x}{x - \sin x}.$$

  a. Find $\lim_{x\to 0} f(x)$.
  b. Use four-digit rounding arithmetic to evaluate $f(0.1)$.
  c. Replace each trigonometric function with its third Maclaurin polynomial and repeat part (b).
  d. The actual value is $f(0.1) = -1.99899998$. Find the relative error for the values obtained in parts (b) and (c).

14. Let

$$f(x) = \frac{e^x - e^{-x}}{x}.$$

  a. Find $\lim_{x\to 0}(e^x - e^{-x})/x$.
  b. Use three-digit rounding arithmetic to evaluate $f(0.1)$.
  c. Replace each exponential function with its third Maclaurin polynomial and repeat part (b).
  d. The actual value is $f(0.1) = 2.003335000$. Find the relative error for the values obtained in parts (b) and (c).

15. Use four-digit rounding arithmetic and the formulas (1.1), (1.2), and (1.3) to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

  a. $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$
  b. $\frac{1}{3}x^2 + \frac{123}{4}x - \frac{1}{6} = 0$
  c. $1.002x^2 - 11.01x + 0.01265 = 0$
  d. $1.002x^2 + 11.01x + 0.01265 = 0$

16. Use four-digit rounding arithmetic and the formulas (1.1), (1.2), and (1.3) to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

  a. $x^2 - \sqrt{7}x + \sqrt{2} = 0$
  b. $\pi x^2 + 13x + 1 = 0$
  c. $x^2 + x - e = 0$
  d. $x^2 - \sqrt{35}x - 2 = 0$

17. Repeat Exercise 15 using four-digit chopping arithmetic.

18. Repeat Exercise 16 using four-digit chopping arithmetic.

19. Use the 64-bit-long real format to find the decimal equivalent of the following floating-point machine numbers.

  a. 0 10000001010 1001001100000000000000000000000000000000000000000000
  b. 1 10000001010 1001001100000000000000000000000000000000000000000000
  c. 0 01111111111 0101001100000000000000000000000000000000000000000000
  d. 0 01111111111 0101001100000000000000000000000000000000000000000001

20. Find the next largest and smallest machine numbers in decimal form for the numbers given in Exercise 19.

APPLIED EXERCISES

26. The opening example to this chapter described a physical experiment involving the temperature of a gas under pressure. In this application, we were given $P = 1.00$ atm, $V = 0.100$ m^3, $N = 0.00420$ mol, and $R = 0.08206$. Solving for T in the ideal gas law gives

$$T = \frac{PV}{NR} = \frac{(1.00)(0.100)}{(0.00420)(0.08206)} = 290.15\text{ K} = 17°\text{C}.$$

In the laboratory, it was found that T was 15°C under these conditions, and when the pressure was doubled and the volume halved, T was 19°C. Assume that the data are rounded values accurate to the places given, and show that both laboratory figures are within the bounds of accuracy for the ideal gas law.

THEORETICAL EXERCISES

27. The binomial coefficient

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!}$$

describes the number of ways of choosing a subset of k objects from a set of m elements.

a. Suppose decimal machine numbers are of the form

$$\pm 0.d_1d_2d_3d_4 \times 10^n, \quad \text{with } 1 \le d_1 \le 9,\ 0 \le d_i \le 9,$$

$$\text{if } i = 2, 3, 4 \quad \text{and} \quad |n| \le 15.$$

What is the largest value of m for which the binomial coefficient $\binom{m}{k}$ can be computed for all k by the definition without causing overflow?

b. Show that $\binom{m}{k}$ can also be computed by

$$\binom{m}{k} = \left(\frac{m}{k}\right)\left(\frac{m-1}{k-1}\right)\cdots\left(\frac{m-k+1}{1}\right).$$

c. What is the largest value of m for which the binomial coefficient $\binom{m}{3}$ can be computed by the formula in part (b) without causing overflow?

d. Use the equation in (b) and four-digit chopping arithmetic to compute the number of possible five-card hands in a 52-card deck. Compute the actual and relative errors.

28. Suppose that $fl(y)$ is a k-digit rounding approximation to y. Show that

$$\left|\frac{y - fl(y)}{y}\right| \le 0.5 \times 10^{-k+1}.$$

[*Hint:* If $d_{k+1} < 5$, then $fl(y) = 0.d_1d_2\ldots d_k \times 10^n$. If $d_{k+1} \ge 5$, then $fl(y) = 0.d_1d_2\ldots d_k \times 10^n + 10^{n-k}$.]

29. Let $f \in C[a, b]$ be a function whose derivative exists on (a, b). Suppose f is to be evaluated at x_0 in (a, b), but instead of computing the actual value $f(x_0)$, the approximate value, $\tilde{f}(x_0)$, is the actual value of f at $x_0 + \epsilon$; that is, $\tilde{f}(x_0) = f(x_0 + \epsilon)$.

a. Use the Mean Value Theorem 1.8 to estimate the absolute error $|f(x_0) - \tilde{f}(x_0)|$ and the relative error $|f(x_0) - \tilde{f}(x_0)|/|f(x_0)|$, assuming $f(x_0) \ne 0$.

b. If $\epsilon = 5 \times 10^{-6}$ and $x_0 = 1$, find bounds for the absolute and relative errors for

i. $f(x) = e^x$

ii. $f(x) = \sin x$

c. Repeat part (b) with $\epsilon = (5 \times 10^{-6})x_0$ and $x_0 = 10$.