

Numerical Analysis I

Numerical Differentiation and Integration

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¹These slides are based on Prof. Tsung-Ming Huang(NTNU)'s original slides

Outline

- 1 Numerical Differentiation
- 2 Richardson Extrapolation Method
- 3 Elements of Numerical Integration
- 4 Composite Numerical Integration
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Numerical Differentiation

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Question

How accurate is

$$\frac{f(x_0 + h) - f(x_0)}{h}?$$

Suppose a given function f has continuous first derivative and f'' exists. From Taylor's theorem

$$f(x + h) = f(x) + f'(x)h + \frac{1}{2}f''(\xi)h^2,$$

where ξ is between x and $x + h$, one has

$$f'(x) = \frac{f(x + h) - f(x)}{h} - \frac{h}{2}f''(\xi) = \frac{f(x + h) - f(x)}{h} + O(h).$$



Hence it is reasonable to use the approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

which is called forward finite difference, and the error involved is

$$|e| = \frac{h}{2} |f''(\xi)| \leq \frac{h}{2} \max_{t \in (x, x+h)} |f''(t)|.$$

Similarly one can derive the backward finite difference approximation

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \tag{1}$$

which has the same order of truncation error as the forward finite difference scheme.



The forward difference is an $O(h)$ scheme. An $O(h^2)$ scheme can also be derived from the Taylor's theorem

$$\begin{aligned}f(x+h) &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(\xi_1)h^3 \\f(x-h) &= f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(\xi_2)h^3,\end{aligned}$$

where ξ_1 is between x and $x+h$ and ξ_2 is between x and $x-h$. Hence

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{1}{6}[f'''(\xi_1) + f'''(\xi_2)]h^3$$

and

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{12}[f'''(\xi_1) + f'''(\xi_2)]h^2$$

Let

$$M = \max_{z \in [x-h, x+h]} f'''(z) \quad \text{and} \quad m = \min_{z \in [x-h, x+h]} f'''(z).$$



If f''' is continuous on $[x - h, x + h]$, then by the intermediate value theorem, there exists $\xi \in [x - h, x + h]$ such that

$$f'''(\xi) = \frac{1}{2}[f'''(\xi_1) + f'''(\xi_2)].$$

Hence

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6}f'''(\xi)h^2 = \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

This is called center difference approximation and the truncation error is

$$|e| = \frac{h^2}{6}f'''(\xi)$$

Similarly, we can derive an $O(h^2)$ scheme from Taylor's theorem for $f''(x)$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{1}{12}f^{(4)}(\xi)h^2,$$

where ξ is between $x - h$ and $x + h$.



Polynomial Interpolation Method

Suppose that $(x_0, f(x_0)), (x_1, f(x_1)) \cdots, (x_n, f(x_n))$ have been given, we apply the Lagrange polynomial interpolation scheme to derive

$$P(x) = \sum_{i=0}^n f(x_i)L_i(x),$$

where

$$L_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j}.$$

Since $f(x)$ can be written as

$$f(x) = \sum_{i=0}^n f(x_i)L(x) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)w(x),$$

where

$$w(x) = \prod_{j=0}^n (x - x_j),$$



we have,

$$\begin{aligned} f'(x) &= \sum_{i=0}^n f(x_i)L'_i(x) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)w'(x) \\ &+ \frac{1}{(n+1)!} w(x) \frac{d}{dx} f^{(n+1)}(\xi_x). \end{aligned}$$

Note that

$$w'(x) = \sum_{j=0}^n \prod_{i=0, i \neq j}^n (x - x_i).$$

Hence a reasonable approximation for the first derivative of f is

$$f'(x) \approx \sum_{i=0}^n f(x_i)L'_i(x).$$

When $x = x_k$ for some $0 \leq k \leq n$,

$$w(x_k) = 0 \quad \text{and} \quad w'(x_k) = \prod_{i=0, i \neq k}^n (x_k - x_i).$$



Hence

$$f'(x_k) = \sum_{i=0}^n f(x_i)L'_i(x_k) + \frac{1}{(n+1)!}f^{(n+1)}(\xi_x) \prod_{i=0, i \neq k}^n (x_k - x_i), \quad (2)$$

which is called an $(n+1)$ -point formula to approximate $f'(x)$.

- Three Point Formulas

Since

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

we have

$$L'_0(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)}.$$

Similarly,

$$L'_1(x) = \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \quad \text{and} \quad L'_2(x) = \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}.$$



Hence

$$\begin{aligned} f'(x_j) &= f(x_0) \left[\frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[\frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right] \\ &+ f(x_2) \left[\frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{k=0, k \neq j}^2 (x_j - x_k), \end{aligned}$$

for each $j = 0, 1, 2$. Assume that

$$x_1 = x_0 + h \text{ and } x_2 = x_0 + 2h, \text{ for some } h \neq 0.$$

Then

$$f'(x_0) = \frac{1}{h} \left[-\frac{3}{2}f(x_0) + 2f(x_1) - \frac{1}{2}f(x_2) \right] + \frac{h^2}{3} f^{(3)}(\xi_0),$$

$$f'(x_1) = \frac{1}{h} \left[-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_2) \right] - \frac{h^2}{6} f^{(3)}(\xi_1),$$

$$f'(x_2) = \frac{1}{h} \left[\frac{1}{2}f(x_0) - 2f(x_1) + \frac{3}{2}f(x_2) \right] + \frac{h^2}{3} f^{(3)}(\xi_2).$$



That is

$$f'(x_0) = \frac{1}{h} \left[-\frac{3}{2}f(x_0) + 2f(x_0 + h) - \frac{1}{2}f(x_0 + 2h) \right] + \frac{h^2}{3}f^{(3)}(\xi_0),$$

$$f'(x_0 + h) = \frac{1}{h} \left[-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_0 + 2h) \right] - \frac{h^2}{6}f^{(3)}(\xi_1), \quad (3)$$

$$f'(x_0 + 2h) = \frac{1}{h} \left[\frac{1}{2}f(x_0) - 2f(x_0 + h) + \frac{3}{2}f(x_0 + 2h) \right] + \frac{h^2}{3}f^{(3)}(\xi_2) \quad (4)$$

Using the variable substitution x_0 for $x_0 + h$ and $x_0 + 2h$ in (3) and (4), respectively, we have

$$f'(x_0) = \frac{1}{2h} [-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + \frac{h^2}{3}f^{(3)}(\xi_0), \quad (5)$$

$$f'(x_0) = \frac{1}{2h} [-f(x_0 - h) + f(x_0 + h)] - \frac{h^2}{6}f^{(3)}(\xi_1),$$

$$f'(x_0) = \frac{1}{2h} [f(x_0 - 2h) - 4f(x_0 - h) + 3f(x_0)] + \frac{h^2}{3}f^{(3)}(\xi_2). \quad (6)$$

Note that (6) can be obtained from (5) by replacing h with $-h$.



- Five-point Formulas

$$f'(x_0) = \frac{1}{12h} [f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] + \frac{h^4}{30} f^{(5)}(\xi),$$

where $\xi \in (x_0 - 2h, x_0 + 2h)$ and

$$f'(x_0) = \frac{1}{12h} [-25f(x_0) + 48f(x_0 + h) - 36f(x_0 + 2h) + 16f(x_0 + 3h) - 3f(x_0 + 4h)] + \frac{h^4}{5} f^{(5)}(\xi),$$

where $\xi \in (x_0, x_0 + 4h)$.



Round-off Error

Consider

$$f'(x_0) = \frac{1}{2h} [-f(x_0 - h) + f(x_0 + h)] - \frac{h^2}{6} f^{(3)}(\xi_1),$$

where $\frac{h^2}{6} f^{(3)}(\xi_1)$ is called truncation error. Let $\tilde{f}(x_0 + h)$ and $\tilde{f}(x_0 - h)$ be the computed values of $f(x_0 + h)$ and $f(x_0 - h)$, respectively. Then

$$f(x_0 + h) = \tilde{f}(x_0 + h) + e(x_0 + h)$$

and

$$f(x_0 - h) = \tilde{f}(x_0 - h) + e(x_0 - h).$$

Therefore, the total error in the approximation

$$f'(x_0) - \frac{\tilde{f}(x_0 + h) - \tilde{f}(x_0 - h)}{2h} = \frac{e(x_0 + h) - e(x_0 - h)}{2h} - \frac{h^2}{6} f^{(3)}(\xi_1)$$

is due in part to round-off error and in part to truncation error.



Assume that

$$|e(x_0 \pm h)| \leq \varepsilon \quad \text{and} \quad |f^{(3)}(\xi_1)| \leq M.$$

Then

$$\left| f'(x_0) - \frac{\tilde{f}(x_0 + h) - \tilde{f}(x_0 - h)}{2h} \right| \leq \frac{\varepsilon}{h} + \frac{h^2}{6} M \equiv e(h).$$

Note that $e(h)$ attains its minimum at $h = \sqrt[3]{3\varepsilon/M}$.

In double precision arithmetics, for example, $\varepsilon \approx |f(x_0 \pm h)| \times 10^{-16}$. The minimum is $O(\sqrt[3]{M\varepsilon^2}) = O(10^{-10})$.



Richardson's Extrapolation

Suppose $\forall h \neq 0$ we have a formula $N_1(h)$ that approximates an unknown value M

$$M - N_1(h) = K_1h + K_2h^2 + K_3h^3 + \dots, \quad (7)$$

for some unknown constants K_1, K_2, K_3, \dots . If $K_1 \neq 0$, then the truncation error is $O(h)$. For example,

$$f'(x) - \frac{f(x+h) - f(x)}{h} = -\frac{f''(x)}{2!}h - \frac{f'''(x)}{3!}h^2 - \frac{f^{(4)}(x)}{4!}h^3 - \dots$$

Goal

Find an easy way to produce formulas with a higher-order truncation error.

Replacing h in (7) by $h/2$, we have

$$M = N_1\left(\frac{h}{2}\right) + K_1\frac{h}{2} + K_2\frac{h^2}{4} + K_3\frac{h^3}{8} + \dots$$



Subtracting (7) with twice (8), we get

$$M = N_2(h) - \frac{K_2}{2}h^2 - \frac{3K_3}{4}h^3 - \dots, \quad (9)$$

where

$$N_2(h) = 2N_1\left(\frac{h}{2}\right) - N_1(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right],$$

which is an $O(h^2)$ approximation formula.

Replacing h in (9) by $h/2$, we get

$$M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{8}h^2 - \frac{3K_3}{32}h^3 - \dots. \quad (10)$$

Subtracting (9) from 4 times (10) gives

$$3M = 4N_2\left(\frac{h}{2}\right) - N_2(h) + \frac{3K_3}{8}h^3 + \dots,$$

which implies that

$$M = \left[N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{3} \right] + \frac{K_3}{8}h^3 + \dots \equiv N_3(h) + \frac{K_3}{8}h^3 + \dots$$



Using induction, M can be approximated by

$$M = N_m(h) + O(h^m),$$

where

$$N_m(h) = N_{m-1}\left(\frac{h}{2}\right) + \frac{N_{m-1}(h/2) - N_{m-1}(h)}{2^{m-1} - 1}.$$

Centered difference formula. From the Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \dots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \dots$$

we have

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!}f'''(x) + \frac{2h^5}{5!}f^{(5)}(x) + \dots,$$



and, consequently,

$$\begin{aligned} f'(x_0) &= \frac{f(x_0 + h) - f(x_0 - h)}{2h} - \left[\frac{h^2}{3!} f'''(x_0) + \frac{h^4}{5!} f^{(5)}(x_0) + \dots \right], \\ &\equiv N_1(h) - \left[\frac{h^2}{3!} f'''(x_0) + \frac{h^4}{5!} f^{(5)}(x_0) + \dots \right]. \end{aligned} \quad (11)$$

Replacing h in (11) by $h/2$ gives

$$f'(x_0) = N_1\left(\frac{h}{2}\right) - \frac{h^2}{24} f'''(x_0) - \frac{h^4}{1920} f^{(5)}(x_0) - \dots \quad (12)$$

Subtracting (11) from 4 times (12) gives

$$f'(x_0) = N_2(h) + \frac{h^4}{480} f^{(5)}(x_0) + \dots,$$

where

$$N_2(h) = \frac{1}{3} \left[4N_1\left(\frac{h}{2}\right) - N_1(h) \right] = N_1\left(\frac{h}{2}\right) + \frac{N_1(h/2) - N_1(h)}{3}.$$



In general,

$$f'(x_0) = N_j(h) + O(h^{2j})$$

with

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}.$$

Example

Suppose that $x_0 = 2.0$, $h = 0.2$ and $f(x) = xe^x$. Compute an approximated value of $f'(2.0) = 22.16716829679195$ to six decimal places.

Solution. By centered difference formula, we have

$$N_1(0.2) = \frac{f(2.0 + 0.2) - f(2.0 - 0.2)}{2h} = 22.414160,$$

$$N_1(0.1) = \frac{f(2.0 + 0.1) - f(2.0 - 0.1)}{h} = 22.228786.$$



It implies that

$$N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = 22.166995$$

which does not have six decimal digits. Adding $N_1(0.05) = 22.182564$, we get

$$N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3} = 22.167157$$

and

$$N_3(0.2) = N_2(0.1) + \frac{N_2(0.1) - N_2(0.2)}{15} = 22.167168$$

which contains six decimal digits.



$O(h)$	$O(h^2)$	$O(h^3)$	$O(h^4)$
1: $N_1(h) = N(h)$			
2: $N_1(h/2) = N(h/2)$	3: $N_2(h)$		
4: $N_1(h/4) = N(h/4)$	5: $N_2(h/2)$	6: $N_3(h)$	
7: $N_1(h/8) = N(h/8)$	8: $N_2(h/4)$	9: $N_3(h/2)$	10: $N_4(h)$



Remark

In practice, we are often encountered with the situation where the order of the numerical method is unknown. That is, the error expansion is of the form

$$M - N(h) = K_1 h^{p_1} + K_2 h^{p_2} + K_3 h^{p_3} + \dots, \quad (13)$$

where p_1, p_2, \dots are unknown. Solving for the leading order p_1 , together with the primary unknowns M and K_1 , requires 3 equations, which can be obtained from, for example, the numerical results at h , $h/2$ and $h/4$:

$$\begin{aligned} M - N(h) &= K_1 h^{p_1} + \dots, \\ M - N\left(\frac{h}{2}\right) &= K_1 \left(\frac{h}{2}\right)^{p_1} + \dots, \\ M - N\left(\frac{h}{4}\right) &= K_1 \left(\frac{h}{4}\right)^{p_1} + \dots \end{aligned} \quad (14)$$

The answer is given by

$$p_1 \approx \log_2 \frac{N(h) - N\left(\frac{h}{2}\right)}{N\left(\frac{h}{2}\right) - N\left(\frac{h}{4}\right)}$$

Once p_1 is known, Richardson extrapolation can be proceeded as before.



Elements of Numerical Integration

The basic method involved in approximating the integration

$$\int_a^b f(x) dx, \quad (15)$$

is called numerical quadrature and uses a sum of the type

$$\int_a^b f(x) dx \approx \sum_{i=0}^n c_i f(x_i). \quad (16)$$

The method of quadrature in this section is based on the polynomial interpolation. We first select a set of distinct nodes $\{x_0, x_1, \dots, x_n\}$ from the interval $[a, b]$. Then the Lagrange polynomial

$$P_n(x) = \sum_{i=0}^n f(x_i) L_i(x) = \sum_{i=0}^n f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

is used to approximate $f(x)$. With the error term we have



$$f(x) = P_n(x) + E_n(x) = \sum_{i=0}^n f(x_i)L_i(x) + \frac{f^{(n+1)}(\zeta_x)}{(n+1)!} \prod_{i=0}^n (x - x_i),$$

where $\zeta_x \in [a, b]$ and depends on x , and

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b P_n(x) dx + \int_a^b E_n(x) dx \\ &= \sum_{i=0}^n f(x_i) \int_a^b L_i(x) dx + \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\zeta_x) \prod_{i=0}^n (x - x_i) dx \end{aligned}$$

The quadrature formula is, therefore,

$$\int_a^b f(x) dx \approx \int_a^b P_n(x) dx = \sum_{i=0}^n f(x_i) \int_a^b L_i(x) dx \equiv \sum_{i=0}^n c_i f(x_i), \quad (18)$$

where

$$c_i = \int_a^b L_i(x) dx = \int_a^b \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} dx.$$



Moreover, the error in the quadrature formula is given by

$$E = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\zeta_x) \prod_{i=0}^n (x - x_i) dx, \quad (20)$$

for some $\zeta_x \in [a, b]$.

Let us consider formulas produced by using first and second Lagrange polynomials with equally spaced nodes. This gives the **Trapezoidal rule** and **Simpson's rule**, respectively.

Trapezoidal rule: Let $x_0 = a, x_1 = b, h = b - a$ and use the linear Lagrange polynomial:

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$

Then

$$\begin{aligned} \int_a^b f(x) dx &= \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) \right] dx \\ &+ \frac{1}{2} \int_{x_0}^{x_1} f''(\zeta(x)) (x - x_0)(x - x_1) dx. \end{aligned} \quad (21)$$



Theorem (Weighted Mean Value Theorem for Integrals)

Suppose $f \in C[a, b]$, the Riemann integral of $g(x)$

$$\int_a^b g(x)dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n g(x_i)\Delta x_i,$$

exists and $g(x)$ does not change sign on $[a, b]$. Then $\exists c \in (a, b)$ with

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

Since $(x - x_0)(x - x_1)$ does not change sign on $[x_0, x_1]$, by the Weighted Mean Value Theorem, $\exists \zeta \in (x_0, x_1)$ such that

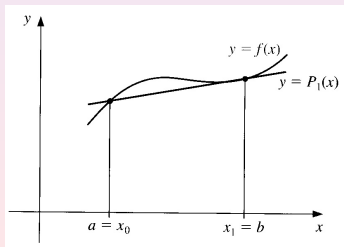
$$\begin{aligned} \int_{x_0}^{x_1} f''(\zeta(x))(x - x_0)(x - x_1)dx &= f''(\zeta) \int_{x_0}^{x_1} (x - x_0)(x - x_1)dx \\ &= f''(\zeta) \left[\frac{x^3}{3} - \frac{x_1 + x_0}{2}x^2 + x_0x_1x \right]_{x_0}^{x_1} = -\frac{h^3}{6} f''(\zeta). \end{aligned}$$



Consequently, Eq. (21) implies that

$$\begin{aligned}\int_a^b f(x)dx &= \left[\frac{(x-x_1)^2}{2(x_0-x_1)} f(x_0) + \frac{(x-x_0)^2}{2(x_1-x_0)} f(x_1) \right]_{x_0}^{x_1} - \frac{h^3}{12} f''(\zeta) \\ &= \frac{x_1-x_0}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\zeta) \\ &= \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\zeta),\end{aligned}$$

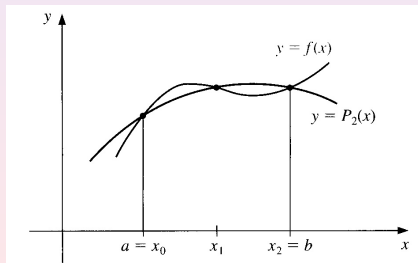
which is called the Trapezoidal rule.



If we choose $x_0 = a$, $x_1 = \frac{1}{2}(a + b)$, $x_2 = b$, $h = (b - a)/2$, and the second order Lagrange polynomial

$$P_2(x) = f(x_0) \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1) \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f(x_2) \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

to interpolate $f(x)$, then



$$\begin{aligned}
 \int_a^b f(x)dx &= \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) \right. \\
 &\quad \left. + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx \\
 &\quad + \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)(x-x_2)}{6} f^{(3)}(\zeta(x)) dx.
 \end{aligned}$$

Since, letting $x = x_0 + th$,

$$\begin{aligned}
 \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} dx &= h \int_0^2 \frac{t-1}{0-1} \cdot \frac{t-2}{0-2} dt \\
 &= \frac{h}{2} \int_0^2 (t^2 - 3t + 2) dt = \frac{h}{3},
 \end{aligned}$$

$$\begin{aligned}
 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} dx &= h \int_0^2 \frac{t-0}{1-0} \cdot \frac{t-2}{1-2} dt \\
 &= -h \int_0^2 (t^2 - 2t) dt = \frac{4h}{3},
 \end{aligned}$$



$$\begin{aligned} \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} dx &= h \int_0^2 \frac{t-0}{2-0} \cdot \frac{t-1}{2-1} dt \\ &= \frac{h}{2} \int_0^2 (t^2 - t) dt = \frac{h}{3}, \end{aligned}$$

it implies that

$$\begin{aligned} \int_a^b f(x) dx &= h \left[\frac{1}{3} f(x_0) + \frac{4}{3} f(x_1) + \frac{1}{3} f(x_2) \right] \\ &\quad + \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)(x-x_2)}{6} f^{(3)}(\zeta(x)) dx, \end{aligned}$$

which is called the Simpson's rule and provides only an $O(h^4)$ error term involving $f^{(3)}$. A higher order error analysis can be derived by expanding f in the third Taylor's formula about x_1 . $\forall x \in [a, b], \exists \zeta_x \in (a, b)$ such that

$$\begin{aligned} f(x) &= f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2 \\ &\quad + \frac{f'''(x_1)}{6}(x-x_1)^3 + \frac{f^{(4)}(\zeta_x)}{24}(x-x_1)^4. \end{aligned}$$



Then

$$\int_a^b f(x) dx = \left[f(x_1)(x - x_1) + \frac{f'(x_1)}{2}(x - x_1)^2 + \frac{f''(x_1)}{6}(x - x_1)^3 + \frac{f'''(x_1)}{24}(x - x_1)^4 \right] \Big|_a^b + \frac{1}{24} \int_a^b f^{(4)}(\zeta_x)(x - x_1)^4 dx.$$

Note that $(b - x_1) = h$, $(a - x_1) = -h$, and since $(x - x_1)^4$ does not change sign in $[a, b]$, by the Weighted Mean-Value Theorem for Integral, there exists $\xi_1 \in (a, b)$ such that

$$\int_a^b f^{(4)}(\zeta_x)(x - x_1)^4 dx = f^{(4)}(\xi_1) \int_a^b (x - x_1)^4 dx = \frac{2f^{(4)}(\xi_1)}{5} h^5.$$

Consequently,

$$\int_a^b f(x) dx = 2f(x_1)h + \frac{f''(x_1)}{3}h^3 + \frac{f^{(4)}(\xi_1)}{60}h^5.$$



Finally we replace $f''(x_1)$ by the central finite difference formulation

$$f''(x_1) = \frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} - \frac{f^{(4)}(\xi_2)}{12}h^2,$$

for some $\xi_2 \in (a, b)$, to obtain

$$\begin{aligned} \int_a^b f(x) dx &= 2hf(x_1) + \frac{h}{3}(f(x_0) - 2f(x_1) + f(x_2)) \\ &\quad - \frac{f^{(4)}(\xi_2)}{36}h^5 + \frac{f^{(4)}(\xi_1)}{60}h^5 \\ &= h \left[\frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{1}{3}f(x_2) \right] \\ &\quad + \frac{1}{90} \left[\frac{3}{2}f^{(4)}(\xi_1) - \frac{5}{2}f^{(4)}(\xi_2) \right] h^5. \end{aligned}$$

It can show that there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{f^{(4)}(\xi)}{90}h^5.$$

This gives the **Simpson's rule formulation**.



Definition

The degree of accuracy, or precision, of a quadrature formula is the largest positive integer p such that the formula is exact for x^k , when $k = 0, 1, \dots, p$.

- The Trapezoidal and Simpson's rules have degrees of precision 1 and 3, respectively.
- The degree of accuracy of a quadrature formula is p if and only if the error $E = 0$ for all polynomials $P(x)$ of degree less than or equal to p , but $E \neq 0$ for some polynomials of degree $p + 1$.



Newton-Cotes Formulas

Definition (Newton-Cotes formula)

A quadrature formula of the form

$$\int_a^b f(x)dx \approx \sum_{i=0}^n c_i f(x_i)$$

is called a Newton-Cotes formula if the nodes $\{x_0, x_1, \dots, x_n\}$ are equally spaced.

Consider a uniform partition of the closed interval $[a, b]$ by

$$x_i = a + ih, \quad i = 0, 1, \dots, n, \quad h = \frac{b - a}{n},$$

where n is a positive integer and h is called the step length.



By introduction a new variable t such that $x = a + ht$, the fundamental Lagrange polynomial becomes

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{a + ht - a - jh}{a + ih - a - jh} = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{t - j}{i - j} \equiv \varphi_i(t).$$

Therefore, the integration (19) gives

$$c_i = \int_a^b L_i(x) dx = \int_0^n \varphi_i(t) h dt = h \int_0^n \prod_{\substack{j=0 \\ j \neq i}}^n \frac{t - j}{i - j} dt, \quad (22)$$

and the general Newton-Cotes formula has the form

$$\int_a^b f(x) dx = h \sum_{i=0}^n f(x_i) \int_0^n \prod_{\substack{j=0 \\ j \neq i}}^n \frac{t - j}{i - j} dt + \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\zeta_x) \prod_{i=0}^n (x - x_i) dx$$



Theorem (Closed Newton-Cotes Formulas)

Suppose that $\sum_{i=0}^n a_i f(x_i)$ denotes the $(n+1)$ -point closed Newton-Cotes formula with $x_0 = a$, $x_n = b$ and $h = (b-a)/n$. If n is even and $f \in C^{n+2}[a, b]$, then

$$\int_a^b f(x) dx = h \sum_{i=0}^n \alpha_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2(t-1)\cdots(t-n) dt, \quad (24)$$

and if n is odd and $f \in C^{n+1}[a, b]$, then

$$\int_a^b f(x) dx = h \sum_{i=0}^n \alpha_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1)\cdots(t-n) dt, \quad (25)$$

where $\xi \in (a, b)$ and $\alpha_i = \int_0^n \prod_{j=0, j \neq i}^n \frac{t-j}{i-j} dt$ for $i = 0, 1, \dots, n$.

Consequently, the degree of accuracy is $n+1$ when n is an even integer, and n when n is an odd integer.

- $n = 1$: Trapezoidal rule

$$\int_a^b f(x) dx = \frac{b-a}{2} [f(a) + f(b)] - \frac{h^3}{12} f''(\xi), \quad a < \xi < b.$$

- $n = 2$: Simpson's rule

$$\int_a^b f(x) dx = h \left[\frac{1}{3} f(x_0) + \frac{4}{3} f(x_1) + \frac{1}{3} f(x_2) \right] - \frac{f^{(4)}(\xi)}{90} h^5, \quad a < \xi < b.$$

- The error term of the Trapezoidal rule is $O(h^3)$.
- Since the rule involves f'' , it gives the exact result when applied to any function whose second derivative is identically zero, e.g., any polynomial of degree 1 or less.
- The degree of accuracy of Trapezoidal rule is one.
- The Simpson's rule is an $O(h^5)$ scheme and the degree of accuracy is three.



Another class of Newton-Cotes formulas is the open Newton-Cotes formulas in which the nodes

$$x_i = x_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$x_0 = a + h \quad \text{and} \quad h = \frac{b - a}{n + 2},$$

are used. This implies that $x_n = b - h$, and the endpoints, a and b , are not used. Hence we label $a = x_{-1}$ and $b = x_{n+1}$. The formulas become

$$\int_a^b f(x)dx = \int_{x_{-1}}^{x_{n+1}} f(x)dx \approx \sum_{i=0}^n a_i f(x_i),$$

where

$$a_i = \int_a^b L_i(x)dx.$$

The following theorem summarizes the open Newton-Cotes formulas.



Theorem (Open Newton-Cotes Formulas)

Suppose that $\sum_{i=0}^n a_i f(x_i)$ denotes the $(n+1)$ -point open Newton-Cotes formula with $x_{-1} = a$, $x_{n+1} = b$ and $h = (b-a)/(n+2)$. If n is even and $f \in C^{n+2}[a, b]$, then

$$\int_a^b f(x) dx = h \sum_{i=0}^n \alpha_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_{-1}^{n+1} t^2(t-1)\cdots(t-n) dt, \quad (26)$$

and if n is odd and $f \in C^{n+1}[a, b]$, then

$$\int_a^b f(x) dx = h \sum_{i=0}^n \alpha_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{n+1} t(t-1)\cdots(t-n) dt, \quad (27)$$

where $\xi \in (a, b)$ and $\alpha_i = \int_{-1}^{n+1} \prod_{j=0, j \neq i}^n \frac{t-j}{i-j} dt$ for $i = 0, 1, \dots, n$.

Consequently, the degree of accuracy is $n+1$ when n is an even integer, and n when n is an odd integer.

The simplest open Newton-Cotes formula is choosing $n = 0$ and only using the midpoint $x_0 = \frac{a+b}{2}$. Then the coefficient and the error term can be computed easily as

$$\alpha_0 = \int_{-1}^1 dt = 2, \quad \text{and} \quad \frac{h^3 f''(\xi)}{2!} \int_{-1}^1 t^2 dt = \frac{1}{3} f''(\xi) h^3.$$

These gives the so-called Midpoint rule or Rectangular rule.
Midpoint Rule:

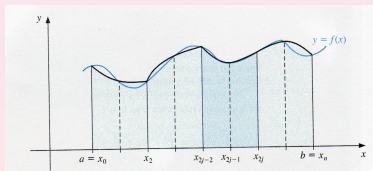
$$\int_a^b f(x) dx = 2hf(x_0) + \frac{1}{3} f''(\xi) h^3 = (b-a)f\left(\frac{a+b}{2}\right) + \frac{1}{3} f''(\xi) h^3, \quad (28)$$

for some $\xi \in (a, b)$.



Composite Numerical Integration

- The Newton-Cotes formulas are generally not suitable for numerical integration over large interval. Higher degree formulas would be required, and the coefficients in these formulas are difficult to obtain.
- Also the Newton-Cotes formulas which are based on polynomial interpolation would be inaccurate over a large interval because of the oscillatory nature of high-degree polynomials.
- Now we discuss a piecewise approach, called composite rule, to numerical integration over large interval that uses the low-order Newton-Cotes formulas.
 - ▶ A composite rule is one obtained by applying an integration formula for a single interval to each subinterval of a partitioned interval.



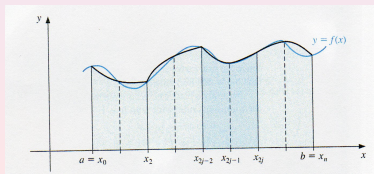
To illustrate the procedure, we choose an even integer n and partition the interval $[a, b]$ into n subintervals by nodes $a = x_0 < x_1 < \cdots < x_n = b$, and apply Simpson's rule on each consecutive pair of subintervals. With

$$h = \frac{b - a}{n} \quad \text{and} \quad x_j = a + jh, \quad j = 0, 1, \dots, n,$$

we have on each interval $[x_{2j-2}, x_{2j}]$,

$$\int_{x_{2j-2}}^{x_{2j}} f(x) dx = \frac{h}{3} [f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})] - \frac{h^5}{90} f^{(4)}(\xi_j),$$

for some $\xi_j \in (x_{2j-2}, x_{2j})$, provided that $f \in C^4[a, b]$.



The composite rule is obtained by summing up over the entire interval, that is,

$$\begin{aligned}
 \int_a^b f(x) dx &= \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x) dx \\
 &= \sum_{j=1}^{n/2} \left[\frac{h}{3} (f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})) - \frac{h^5}{90} f^{(4)}(\xi_j) \right] \\
 &= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2) \\
 &\quad + f(x_2) + 4f(x_3) + f(x_4) \\
 &\quad + f(x_4) + 4f(x_5) + f(x_6) \\
 &\quad \vdots \\
 &\quad + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)
 \end{aligned}$$



Hence

$$\begin{aligned}\int_a^b f(x) dx &= \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) \\ &\quad + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \\ &= \frac{h}{3} \left[f(x_0) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(x_n) \right] \\ &\quad - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j).\end{aligned}$$

To estimate the error associated with approximation, since $f \in C^4[a, b]$, we have, by the Extreme Value Theorem,

$$\min_{x \in [a, b]} f^{(4)}(x) \leq f^{(4)}(\xi_j) \leq \max_{x \in [a, b]} f^{(4)}(x),$$

for each $\xi_j \in (x_{2j-2}, x_{2j})$.



Hence

$$\frac{n}{2} \min_{x \in [a,b]} f^{(4)}(x) \leq \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \frac{n}{2} \max_{x \in [a,b]} f^{(4)}(x),$$

and

$$\min_{x \in [a,b]} f^{(4)}(x) \leq \frac{2}{n} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \max_{x \in [a,b]} f^{(4)}(x).$$

By the Intermediate Value Theorem, there exists $\mu \in (a, b)$ such that

$$f^{(4)}(\mu) = \frac{2}{n} \sum_{j=1}^{n/2} f^{(4)}(\xi_j).$$

Thus, by replacing $n = (b - a)/h$,

$$\sum_{j=1}^{n/2} f^{(4)}(\xi_j) = \frac{n}{2} f^{(4)}(\mu) = \frac{b-a}{2h} f^{(4)}(\mu).$$

Consequently, the composite Simpson's rule is derived.



Composite Simpson's Rule

$$\int_a^b f(x) dx = \frac{h}{3} \left[f(a) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(b) \right] - \frac{b-a}{180} f^{(4)}(\mu) h^4,$$

where n is an even integer, $h = (b-a)/n$, $x_j = a + jh$, for $j = 0, 1, \dots, n$, and some $\mu \in (a, b)$.

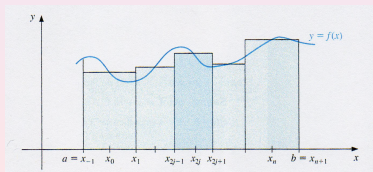


The composite Midpoint rule can be derived in a similar way, except the midpoint rule is applied on each subinterval $[x_{2j-1}, x_{2j+1}]$ instead. That is,

$$\int_{x_{2j-1}}^{x_{2j+1}} f(x) dx = 2hf(x_{2j}) + \frac{h^3}{3}f''(\xi_j), \quad j = 1, 2, \dots, \frac{n}{2}.$$

Note that n must again be even. Consequently,

$$\int_a^b f(x) dx = 2h \sum_{j=1}^{n/2} f(x_{2j}) + \frac{h^3}{3} \sum_{j=1}^{n/2} f''(\xi_j).$$



The error term can be written as

$$\sum_{j=1}^{n/2} f''(\xi_j) = \frac{n}{2} f''(\mu) = \frac{b-a}{2h} f''(\mu),$$

for some $\mu \in (a, b)$.

Composite Midpoint Rule

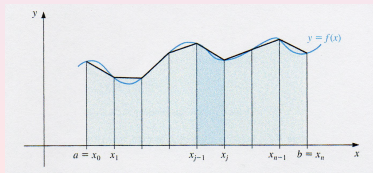
$$\int_a^b f(x) dx = 2h \sum_{j=1}^{n/2} f(x_{2j}) + \frac{b-a}{6} f''(\mu) h^2, \quad (29)$$

where n is an even integer, $h = (b-a)/n$, $x_j = a + jh$, for $j = 0, 1, \dots, n$, and some $\mu \in (a, b)$.



To derive the composite Trapezoidal rule, we partition $[a, b]$ by n equally spaced nodes $a = x_0 < x_1 < \cdots < x_n = b$, where n can be either odd or even. Apply the trapezoidal rule on $[x_{j-1}, x_j]$ and sum them up to obtain

$$\begin{aligned}
 \int_a^b f(x) dx &= \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x) dx \\
 &= \sum_{j=1}^n \left\{ \frac{h}{2} [f(x_{j-1}) + f(x_j)] - \frac{h^3}{12} f''(\xi_j) \right\} \\
 &= \frac{h}{2} \{ [f(x_0) + f(x_1)] + [f(x_1) + f(x_2)] + \cdots \\
 &\quad + [f(x_{n-1}) + f(x_n)] \} - \frac{h^3}{12} \sum_{j=1}^n f''(\xi_j)
 \end{aligned}$$



Hence,

$$\begin{aligned}\int_a^b f(x) dx &= \frac{h}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ &\quad - \frac{h^3}{12} \sum_{j=1}^n f''(\xi_j) \\ &= \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{h^3}{12} \sum_{j=1}^n f''(\xi_j) \\ &= \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} f''(\mu) h^2,\end{aligned}$$

where each $\xi_j \in (x_{j-1}, x_j)$ and $\mu \in (a, b)$.



Composite Trapezoidal Rule

$$\int_a^b f(x) dx = \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} f''(\mu) h^2, \quad (30)$$

where n is an integer, $h = (b-a)/n$, $x_j = a + jh$, for $j = 0, 1, \dots, n$, and some $\mu \in (a, b)$.



Gaussian Quadrature

Newton-Cotes formulas:

- The choice of nodes x_0, x_1, \dots, x_n was made *a priori*.
- Use values of the function at equally spaced points.
- Once the nodes were fixed, the coefficients were determined, e.g., by integrating the fundamental Lagrange polynomials of degree n .
- These formulas are exact for polynomials of degree $\leq n$ ($n + 1$, if n is even).

This approach is convenient when the formulas are combined to form the composite rules, but the restriction may decrease the accuracy of the approximation.



Gaussian quadrature

- 1 Chooses the points for evaluation in an optimal, rather than pre-fixed or equally-spaced, way.
- 2 The nodes $x_0, x_1, \dots, x_n \in [a, b]$ and the coefficients c_0, c_1, \dots, c_n are chosen to minimize the expected error obtained in the approximation

$$\int_a^b f(x) dx \approx \sum_{i=0}^n c_i f(x_i) \quad (31)$$

- 3 Produce the exact result for the largest class of polynomials, that is, the choice which gives the greatest degree of precision.

The coefficients c_0, c_1, \dots, c_n are arbitrary, and the nodes x_0, x_1, \dots, x_n are restricted only in $[a, b]$. These give $2n + 2$ degrees of freedom. Thus we can expect that the quadrature formula of (31) can be discovered that will be exact for polynomials of degree $\leq 2n + 1$.



Suppose we want to determine c_0, c_1, x_0 and x_1 so that

$$\int_{-1}^1 f(x)dx \approx c_0 f(x_0) + c_1 f(x_1) \quad (32)$$

gives the exact result whenever $f(x)$ is a polynomial of degree $2 \times 2 - 1 = 3$ or less, i.e.,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

Since

$$\begin{aligned} & \int (a_0 + a_1x + a_2x^2 + a_3x^3)dx \\ &= a_0 \int 1dx + a_1 \int xdx + a_2 \int x^2dx + a_3 \int x^3dx, \end{aligned}$$

this is equivalent to show that (32) gives exact results when $f(x)$ is $1, x, x^2$ and x^3 . Hence



$$\begin{aligned}c_0 + c_1 &= \int_{-1}^1 1 dx = 2, \\c_0 x_0 + c_1 x_1 &= \int_{-1}^1 x dx = 0, \\c_0 x_0^2 + c_1 x_1^2 &= \int_{-1}^1 x^2 dx = \frac{2}{3}, \\c_0 x_0^3 + c_1 x_1^3 &= \int_{-1}^1 x^3 dx = 0.\end{aligned}$$

It implies that

$$c_0 = 1, \quad c_1 = 1, \quad x_0 = -\frac{\sqrt{3}}{3}, \quad x_1 = \frac{\sqrt{3}}{3}$$

which gives

$$\int_{-1}^1 f(x) dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$



Theorem

Suppose that x_0, x_1, \dots, x_n are the roots of the $(n + 1)$ -st Legendre polynomial p_{n+1} , and that for each $i = 0, 1, \dots, n$,

$$c_i = \int_{-1}^1 \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} dx.$$

If $f(x)$ is any polynomial of degree $\leq 2n + 1$, then

$$\int_{-1}^1 f(x) dx = \sum_{i=0}^n c_i f(x_i).$$

Gaussian Quadrature Rule

$$\int_{-1}^1 f(x) dx \approx \sum_{i=0}^n c_i f(x_i), \quad (33)$$

Orthogonalization and Legendre polynomials

Definition

- 1 In an inner-product space, we say f is orthogonal to g , and write $f \perp g$ if $\langle f, g \rangle = 0$.
- 2 We write $f \perp G$ if $f \perp g$ for all $g \in G$.
- 3 We say that a finite or infinite sequence of vectors f_1, f_2, \dots in an inner-product space is orthogonal if $\langle f_i, f_j \rangle = 0$ for all $i \neq j$, and orthonormal if $\langle f_i, f_j \rangle = \delta_{ij}$.

The space of continuous functions on $[a, b]$ with inner-product defined as

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx, \quad (34)$$

is an inner-product space.



Definition

$\{\phi_0, \phi_1, \dots, \phi_n\}$, where $\phi_i \in C[a, b]$ for all $i = 0, 1, \dots, n$, is said to be an orthogonal set of functions if

$$\langle \phi_i, \phi_j \rangle = \int_a^b \phi_i(x)\phi_j(x) dx = \begin{cases} 0, & \text{when } i \neq j, \\ \alpha_i > 0, & \text{when } i = j. \end{cases}$$

If, in addition, $\alpha_i = 1$ for all i , then the set is said to be orthonormal.



Definition

Legendre polynomials: Gram-Schmidt process applied to $1, x, x^2, \dots$.

$$p_0(x) = 1$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 = x$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1 = x^2 - \frac{1}{3}$$

...



Corollary

For any $n > 0$, the set of Legendre polynomials $\{p_0, p_1, \dots, p_n\}$ defined above is linearly independent and

$$\langle q, p_n \rangle = \int_a^b q(x)p_n(x) dx = 0$$

for any polynomial $q(x)$ with $\deg(q(x)) \leq n - 1$.



Let Π_n denote the set of polynomials of degree at most n , that is,

$$\Pi_n = \{p(x) \mid p(x) \text{ is a polynomial and } \deg(p) \leq n\}.$$

Theorem

Let $q(x)$ be any nonzero polynomial of degree $n + 1$, and $q(x) \perp \Pi_n$. If x_0, x_1, \dots, x_n are the roots of $q(x)$ in $[a, b]$, and

$$c_i = \int_a^b \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} dx,$$

then

$$\int_a^b p(x) dx = \sum_{i=0}^n c_i p(x_i), \quad \text{for any } p \in \Pi_{2n+1}.$$

That is, the quadrature rule is exact for any polynomial of degree $\leq 2n + 1$.



Proof. For any polynomial $p \in \Pi_{2n+1}$, we can write

$$p(x) = q(x)t(x) + r(x),$$

where $t(x), r(x) \in \Pi_n$. Since x_0, x_1, \dots, x_n are roots of $q(x)$, we have

$$p(x_i) = q(x_i)t(x_i) + r(x_i) = r(x_i), \quad i = 0, 1, \dots, n.$$

By assumption, $q \perp \Pi_n$, we have

$$\langle q, t \rangle = \int_a^b q(x)t(x) dx = 0.$$

Since $r(x) \in \Pi_n$, it can be expressed exactly in the Lagrange form

$$r(x) = \sum_{i=0}^n r(x_i) \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}.$$



Hence

$$\begin{aligned}\int_a^b p(x) dx &= \int_a^b q(x)t(x) dx + \int_a^b r(x) dx \\ &= \int_a^b r(x) dx = \int_a^b \sum_{i=0}^n r(x_i) \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} dx \\ &= \sum_{i=0}^n r(x_i) \int_a^b \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} dx \\ &= \sum_{i=0}^n p(x_i) \int_a^b \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} dx \\ &= \sum_{i=0}^n c_i p(x_i).\end{aligned}$$



If the interval $[a, b]$ is $[-1, 1]$, then we can obtain a set of orthogonal polynomials called the Legendre polynomials. The first few Legendre polynomials are

$$p_0(x) = 1$$

$$p_1(x) = x$$

$$p_2(x) = x^2 - \frac{1}{3}$$

$$p_3(x) = x^3 - \frac{3}{5}x$$

$$p_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

$$p_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$$



Gaussian Quadrature Rule

For a given function $f(x) \in C[-1, 1]$ and integer n ,

$$\int_{-1}^1 f(x) dx \approx \sum_{i=0}^n c_i f(x_i), \quad (35)$$

where x_0, x_1, \dots, x_n are the roots of the $(n + 1)$ -st Legendre polynomial p_{n+1} , and

$$c_i = \int_{-1}^1 \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} dx. \quad i = 0, 1, \dots, n.$$



n	x_i	c_i
0	$x_0 = 0$	$c_0 = 2$
1	$x_0 = -0.5773502692$ $x_1 = 0.5773502692$	$c_0 = c_1 = 1$
2	$x_0 = -0.7745966692$ $x_1 = 0$ $x_2 = 0.7745966692$	$c_0 = \frac{5}{9}$ $c_1 = \frac{8}{9}$ $c_2 = \frac{5}{9}$
3	$x_0 = -0.8611363116$ $x_1 = -0.3399810436$ $x_2 = 0.3399810436$ $x_3 = 0.8611363116$	$c_0 = 0.3478548451$ $c_1 = 0.6521451549$ $c_2 = 0.6521451549$ $c_3 = 0.3478548451$
4	$x_0 = -0.9061798459$ $x_1 = -0.5384693101$ $x_2 = 0$ $x_3 = 0.5384693101$ $x_4 = 0.9061798459$	$c_0 = 0.2369268851$ $c_1 = 0.4786286705$ $c_2 = \frac{128}{225} = 0.568888889$ $c_3 = 0.4786286705$ $c_4 = 0.2369268851$



Romberg Integration

Recall the trapezoidal rule integral formulation

$$\begin{aligned}\int_a^b f(x) dx &\approx T(n) \\ &= \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \cdots \\ &\quad + 2f(a+(n-1)h) + f(a+nh)],\end{aligned}$$

where $h = \frac{b-a}{n}$. Let $n = 2$ and $h = \frac{b-a}{2n}$.

• If we only consider the partitions by x_0, x_2 , and x_4 , and apply the Trapezoidal rule, we have the approximation

$$T(n) = \frac{2h}{2}[f(a) + 2f(a+2h) + f(a+4h)] = h[f(a) + 2f(a+2h) + f(a+4h)].$$

• If we apply Trapezoidal rule by x_0, x_1, x_2, x_3 , and x_4 , then we have

$$\begin{aligned}T(2n) &= \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + 2f(a+3h) + f(a+4h)] \\ &= \frac{1}{2}T(n) + h[f(a+h) + f(a+3h)].\end{aligned}$$

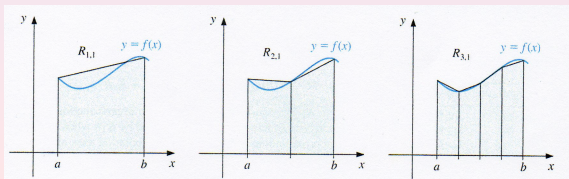


This observation shows that

- if we have computed $T(n)$ and the step size is half, we don't have to compute all the function values all over again, but just at those newly added points.
- In general, suppose $T(n)$ has been computed and the step size becomes $h = \frac{b-a}{2n}$, then

$$T(2n) = \frac{1}{2}T(n) + h \sum_{i=1}^n f(a + (2i - 1)h). \quad (36)$$

With this idea in mind, we can apply the trapezoidal rule recursively, i.e., we partition the interval $[a, b]$ into 2^n subintervals with $n = 1, 2, 3, \dots$,



and the integral formulation becomes

$$T(2^n) = \frac{1}{2}T(2^{n-1}) + \frac{b-a}{2^n} \sum_{i=1}^{2^{n-1}} f\left(a + (2i-1)\frac{b-a}{2^n}\right). \quad (37)$$

Let

$$R_{k,1} = T(2^k).$$

If $f \in C^\infty[a, b]$, then

$$\int_a^b f(x)dx - R_{k,1} = K_1 h_k^2 + \sum_{i=2}^{\infty} K_i h_k^{2^i}, \quad (38)$$

where each K_i is independent of h_k and depends only on $f^{(2^i-1)}(a)$ and $f^{(2^i-1)}(b)$. Replacing h_k with $h_{k+1} = h_k/2$, we get

$$\int_a^b f(x)dx - R_{k+1,1} = \frac{K_1 h_k^2}{4} + \sum_{i=2}^{\infty} \frac{K_i h_k^{2^i}}{4^i}.$$



Subtracting Eq. (38) from 4 times Eq. (39), we have

$$\int_a^b f(x)dx - \left[R_{k+1,1} + \frac{R_{k+1,1} - R_{k,1}}{3} \right] = -\frac{K_2}{4}h_k^4 + \sum_{i=3}^{\infty} \frac{K_i}{3} \left(\frac{1 - 4^{i-1}}{4^{i-1}} \right) h_k^2$$

Define

$$R_{k,2} = R_{k,1} + \frac{R_{k,1} - R_{k-1,1}}{3}, \quad \text{for } k = 2, 3, \dots, n.$$

Apply the extrapolation procedure to those values. Continuing this notation, we have, for each $k = 2, 3, 4, \dots, n$ and $j = 2, \dots, k$, an $O(h_k^{2j})$ approximation formula defined by

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}.$$

This is called the Romberg algorithm.



Algorithm (Romberg Integration Algorithm)

Use the Romberg algorithm to evaluate $\int_a^b f(x) dx$.

$$R(0, 0) = \frac{1}{2}(b - a)[f(a) + f(b)]$$

For $n = 1, 2, \dots, M$

$$R(n, 0) = \frac{1}{2}R(n - 1, 0) + \frac{b-a}{2^n} \sum_{i=1}^{2^{n-1}} f(a + (2i - 1)\frac{b-a}{2^n})$$

End for

For $k = 1, 2, \dots, M$

For $n = k, k + 1, \dots, M$

$$R(n, k) = R(n, k - 1) + \frac{1}{4^{k-1}}[R(n, k - 1) - R(n - 1, k - 1)]$$

End for

End for



Theorem

If $f \in C[a, b]$, then for each column k ,

$$\lim_{n \rightarrow \infty} R(n, k) = \int_a^b f(x) dx. \quad (40)$$

Moreover, if $f \in C^{2m}[a, b]$, then $R(n, m)$ converges to $\int_a^b f(x) dx$ with a rate of $O(h^{2m})$, where $h = \frac{b-a}{2^n}$.

