

Improper integrals

(I) Left endpoint singularity

Example $I = \int_0^1 \frac{1}{\sqrt{x}} dx$

$$I = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{1}{\sqrt{x}} dx$$

$$I_M = \sum_{i=1}^n h \frac{1}{\sqrt{x_{i-\frac{1}{2}}}}$$

where $x_i = ih = \frac{i}{n}$

$$x_{i-\frac{1}{2}} = (i-\frac{1}{2})h = \frac{i-\frac{1}{2}}{n}$$

$$I - I_M = \sum_{i=1}^n \left(\int_{x_{i-1}}^{x_i} \frac{1}{\sqrt{x}} dx - h \frac{1}{\sqrt{x_{i+\frac{1}{2}}}} \right)$$

$$= \sum_{i=1}^n e_i$$

$$e_1 = \int_0^h \frac{1}{\sqrt{x}} dx - h \cdot \frac{1}{\sqrt{\frac{h}{2}}}$$

$$= 2\sqrt{h} - \sqrt{2}\sqrt{h} = (2-\sqrt{2})\sqrt{h}$$

$$e_i = Ch^3 f''(\xi_i)$$

e_1, \dots, e_n are of the same sign!

$$|I - I_M| \geq |e_1| = O(\sqrt{h})$$

Similarly, one can show
that for Simpson's rule

$$|I - I_S| \geq |e_1| = Cn^4$$

No improvement over I_M .

How to de-singularize?

Eg. Simpson's for $\int_0^1 \frac{e^x}{\sqrt{x}} dx$,

(Method A). subtract singular part

write $\frac{e^x}{\sqrt{x}} = f_{\text{sing}} + f_{\text{reg}}$

with $|f_{\text{reg}}^{(4)}| \leq M$

Taylor's expansion gives

$$e^x = \underbrace{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)\right)}_{P_4(x)}$$

$$\begin{aligned}\int_0^1 \frac{e^x}{\sqrt{x}} dx &= \int_0^1 \frac{P_4(x)}{\sqrt{x}} dx + \int_0^1 \frac{e^x - P_4(x)}{\sqrt{x}} dx \\ &= \textcircled{1} + \textcircled{2}\end{aligned}$$

① : exact integration

$$\left(\frac{e^x - P_4(x)}{\sqrt{x}}\right) = O(x^{\frac{9}{2}}) \in C^4([0,1])$$

$$|\textcircled{2} - \textcircled{2}_s| \leq Ch^4$$

$$\underline{A_{hs}} = \textcircled{1} + \textcircled{2}_s$$

Method B. Change of variable.

try $x = y^p$ and find suitable $p > 0$

$$\int_0^1 \frac{e^x}{\sqrt{x}} dx = \int_0^1 \frac{e^{y^p}}{y^{\frac{p}{2}}} \cdot p y^{p-1} dy$$

$$= p \int_0^1 \underbrace{e^{y^p} y^{(\frac{p}{2}-1)}}_{\text{Want } \in C_y^4} dy$$

take $p=2$ will do

(II) right endpoint singularity

$$\int_a^b f(x) dx \quad z = b - x$$

\Rightarrow left endpoint singularity

(III) Infinite Singularity

Example $\int_1^{\infty} x^{-\frac{3}{2}} \sin \frac{1}{x} dx$

change of variables

$$1 \leq x < \infty \iff 0 < t \leq 1$$

Textbook method: $x = \frac{1}{t}$

$$\int_1^{\infty} x^{-\frac{3}{2}} \sin \frac{1}{x} dx = \int_{t=1}^{t=0} t^{\frac{3}{2}} (\sin t) \left(-\frac{1}{t^2} \right) dt$$

\rightarrow desingularize $O(t^{\frac{1}{2}})$ singularity.

Alternative change of variable

$$1 \leq x < \infty \iff 0 < s \leq b$$

try $x = s^{-\frac{1}{\ell}}$

and find suitable $\ell > 0$

$$\int_1^{\infty} x^{-\frac{3}{2}} \sin \frac{1}{x} dx$$

$$= \int_{s=1}^{s=0} \underbrace{s^{\frac{3\ell}{2}} (\sin s^{\frac{1}{\ell}}) (-\frac{1}{\ell}) s^{-\frac{1}{\ell}-1}}_{\text{Want it in } C_s} ds$$

$$= \frac{1}{\ell} \int_{s=0}^1 \underbrace{s^{\frac{\ell}{2}-1} \sin s^{\frac{1}{\ell}}}_{\text{Want it in } C_s} ds$$

take $\ell = 2$ will do

Example: $\int_0^2 \frac{x e^x}{\sqrt[3]{(x-1)}} dx = \int_0^1 + \int_1^2$

\int_0^1 and \int_1^2 evaluated separately, but similarly

$$\int_0^1 \frac{x e^x}{(x-1)^{\frac{1}{3}}} dx \quad z = 1-x$$

$$= \int_{z=1}^0 \frac{(1-z) e^{(1-z)}}{-z^{\frac{1}{3}}} (-dz)$$

$$(z = t^{\frac{1}{3}})$$

$$= - \int_{t=0}^1 \frac{(1-t^{\frac{1}{3}}) e^{(1-t^{\frac{1}{3}})}}{t^{\frac{1}{3}}} \frac{1}{3} t^{(\frac{1}{3}-1)} dt \quad (\text{take } \frac{1}{3} = 3)$$

Want it smooth for $t \in [0,1]$