Higher order approximation of $f'(x_0)$

$$-\gamma \quad f(x_0-2h) = 1 \quad -2h \quad + \frac{(2h)^2}{2} \quad - \frac{(2h)^3}{6} \quad + \frac{(2h)^4}{24} \quad - O(h^5)$$

$$-\beta \quad f(x_0-h) = 1 \quad -h \quad + \frac{h^2}{2} \quad - \frac{h^3}{6} \quad + \frac{h^4}{24} \quad - O(h^5)$$

$$\alpha\,(=0) \quad f(x_0) = f(x_0)$$

$$\beta \quad f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f^{(3)}(x_0) + \frac{h^4}{24} f^{(4)}(x_0) + O(h^5)$$

$$\gamma \quad f(x_0+2h) = 1 + 2h \quad + \frac{(2h)^2}{2} \quad + \frac{(2h)^3}{6} \quad + \frac{(2h)^4}{24} \quad + O(h^5)$$

$$\begin{cases} \alpha = 0 \\ 2\beta h + 4\gamma h = 1 \\ \frac{2\beta}{6} h^3 + \frac{2\gamma}{6}(2h)^3 = 0 \end{cases}$$

0 1 0 error term for $O(h^2)$ approximation

0 1 0 0 0 error term for $O(h^4)$ approximation

① ②

$$f'(x_0) = (-\gamma f(x_0-2h) - \beta f(x_0-h) + \alpha f(x_0) + \beta f(x_0+h) + \gamma f(x_0+2h)) + O(h^4)$$

$$-\gamma \quad f(x_0-2h) = 1 \quad -2h \quad + \frac{(2h)^2}{2} \quad - \frac{(2h)^3}{6} \quad + \frac{(2h)^4}{24} \quad - \frac{(2h)^5}{120} f^{(5)}(\xi_{--})$$

$$-\beta \quad f(x_0-h) = 1 \quad -h \quad + \frac{h^2}{2} \quad - \frac{h^3}{6} \quad + \frac{h^4}{24} \quad - \frac{h^5}{120} f^{(5)}(\xi_-)$$

$$\alpha \quad f(x_0) = f(x_0)$$

$$\beta \quad f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f^{(3)}(x_0) + \frac{h^4}{24} f^{(4)}(x_0) + \frac{h^5}{120} f^{(5)}(\xi_+)$$

$$\gamma \quad f(x_0+2h) = \underbrace{f(x_0)}_{0} + \underbrace{2h f'(x_0)}_{0} + \underbrace{\frac{(2h)^2}{2} f''(x_0)}_{0} + \underbrace{\frac{(2h)^3}{6} f^{(3)}(x_0)}_{1} + \underbrace{\frac{(2h)^4}{24} f^{(4)}(x_0)}_{0} + \underbrace{\frac{(2h)^5}{120} f^{(5)}(\xi_{++})}_{\text{error term}}$$

Centered difference approximation of $f^{(3)}(x_0)$ (2nd order)

$\alpha = 0$

$2\beta h + 4\gamma h = 0$

$\frac{2\beta h^3}{6} + \frac{2\gamma (2h)^3}{6} = 1$

$$\Rightarrow \left(-\gamma f(x_0-2h) - \beta f(x_0-h) + \alpha f(x_0) + \beta f(x_0+h) + \gamma f(x_0+2h)\right) = f^{(3)}(x_0) + O\left(f^{(5)}(\xi) \cdot h^2\right)$$

$f^{(4)}(x_0)$ approximation 0 0 0 0 1 0 by symmetry error term