Error term of first derivative with $O(h^4)$

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2}h^2+\frac{f^{(3)}(x_0)}{6}h^3+\frac{f^{(4)}(x_0)}{24}h^4+\frac{1}{24}\int_{x_0}^{x_0+h} f^{(5)}(t)(x_0+h-t)^4dt$$

$$f(x_0-h) = f(x_0)-f'(x_0)h+\frac{f''(x_0)}{2}h^2-\frac{f^{(3)}(x_0)}{6}h^3+\frac{f^{(4)}(x_0)}{24}h^4+\frac{1}{24}\int_{x_0}^{x_0-h} f^{(5)}(t)(x_0-h-t)^4dt$$

We have

$$f'(x_0) = \frac{1}{2h}[f(x_0+h)-f(x_0-h)] - \frac{f^{(3)}(x_0)}{6}h^2 - \frac{1}{48h}[\int_{x_0}^{x_0+h} f^{(5)}(t)(x_0+h-t)^4dt + \int_{x_0-h}^{x_0} f^{(5)}(t)(x_0-h-t)^4dt]$$

Also,

$$f'(x_0) = \frac{1}{4h}[f(x_0+2h)-f(x_0-2h)] - \frac{f^{(3)}(x_0)}{6}4h^2 - \frac{1}{96h}[\int_{x_0}^{x_0+2h} f^{(5)}(t)(x_0+2h-t)^4dt + \int_{x_0-2h}^{x_0} f^{(5)}(t)(x_0-2h-t)^4dt]$$

which implies that

$$\begin{aligned} f'(x_0) &= \frac{2}{3h}[f(x_0+h)-f(x_0-h)] - \frac{1}{12h}[f(x_0+2h)-f(x_0-2h)] \\ &- \frac{1}{36h}[\int_{x_0}^{x_0+h} f^{(5)}(t)(x_0+h-t)^4dt + \int_{x_0-h}^{x_0} f^{(5)}(t)(x_0-h-t)^4dt] \\ &+ \frac{1}{288h}[\int_{x_0}^{x_0+2h} f^{(5)}(t)(x_0+2h-t)^4dt + \int_{x_0-2h}^{x_0} f^{(5)}(t)(x_0-2h-t)^4dt] \\ &= \frac{1}{12h}[f(x_0-2h)-8f(x_0-h)+8f(x_0+h)-f(x_0+2h)] + \textbf{error term} \end{aligned}$$

where the **error term** is

$$\begin{aligned} &- \frac{1}{36h}[\int_{x_0}^{x_0+h} f^{(5)}(t)(x_0+h-t)^4dt + \int_{x_0-h}^{x_0} f^{(5)}(t)(x_0-h-t)^4dt] \\ &+ \frac{1}{288h}[\int_{x_0}^{x_0+2h} f^{(5)}(t)(x_0+2h-t)^4dt + \int_{x_0-2h}^{x_0} f^{(5)}(t)(x_0-2h-t)^4dt] \end{aligned}$$

By letting $u = x_0+h-t$ and $u = t-x_0+h$, respectively,
the **error term** becomes

$$\begin{aligned} &- \frac{1}{36h}\int_0^h f^{(5)}(x_0+h-u)u^4du + \frac{1}{288h}\int_{-h}^h f^{(5)}(x_0+h-u)(h+u)^4du \\ &- \frac{1}{36h}\int_0^h f^{(5)}(x_0-h+u)u^4du + \frac{1}{288h}\int_{-h}^h f^{(5)}(x_0-h+u)(h+u)^4du \end{aligned}$$

$$= \int_0^h f^{(5)}(x_0+h-u)[-\frac{1}{36h}u^4+\frac{1}{288h}(h+u)^4]du+\int_0^h f^{(5)}(x_0-h+u)[-\frac{1}{36h}u^4+\frac{1}{288h}(h+u)^4]du \qquad (1)$$

$$+\frac{1}{288h}\int_{-h}^0 f^{(5)}(x_0+h-u)(h+u)^4du+\frac{1}{288h}\int_{-h}^0 f^{(5)}(x_0-h+u)(h+u)^4du \qquad (2)$$

$$\begin{aligned}
(1) &= \int_0^h [f^{(5)}(x_0+h-u)+f^{(5)}(x_0-h+u)][-\frac{1}{36h}u^4+\frac{1}{288h}(h+u)^4]du \\
&= \int_0^h 2f^{(5)}(\xi(u))(-\frac{1}{36h})[u^4-\frac{1}{8}(h+u)^4]du \quad \text{(By I.V.T.)} \\
&= 2f^{(5)}(\xi)(-\frac{1}{36h})\int_0^h [u^4-\frac{1}{8}(h+u)^4]du \quad \text{(By M.V.T. Note } [u^4-\frac{1}{8}(h+u)^4]<0,\ \forall u\in[0,h].\text{)} \\
&= \frac{23}{720}f^{(5)}(\xi)h^4
\end{aligned}$$

$$\begin{aligned}
(2) &= \frac{1}{288h}\int_{-h}^0 [f^{(5)}(x_0+h-u)+f^{(5)}(x_0-h+u)](h+u)^4du \\
&= \frac{1}{288h}\int_{-h}^0 [2f^{(5)}(\eta(w))](h+u)^4du \quad \text{(By I.V.T.)} \\
&= \frac{f^{(5)}(\eta)}{144h}\int_{-h}^0 (h+u)^4du \quad \text{(By M.V.T. Note } (h+u)^4>0.\text{)} \\
&= \frac{1}{720}f^{(5)}(\eta)h^4
\end{aligned}$$

Thus, the **error term** is

$$\begin{aligned}
(1)+(2) &= \frac{23}{720}f^{(5)}(\xi)h^4+\frac{1}{720}f^{(5)}(\eta)h^4 \quad \text{(By I.V.T.)} \\
&= \frac{1}{30}f^{(5)}(\xi_1)h^4 \qquad \square
\end{aligned}$$

Note that

$$\begin{aligned}
&u^4-\frac{1}{8}(h+u)^4 \\
&= (u^2+\frac{1}{2^{\frac{3}{2}}}(h+u)^2)\ (u^2-\frac{1}{2^{\frac{3}{2}}}(h+u)^2) \\
&= (u^2+\frac{1}{2^{\frac{3}{2}}}(h+u)^2)\ (u+\frac{1}{2^{\frac{3}{4}}}(h+u)\ )\ (u-\frac{1}{2^{\frac{3}{4}}}(h+u)\ )<0\ \forall u\in[0,h]
\end{aligned}$$