Error term of Simpson's rule

$$f(x_1+h)=f(x_1)+hf^{'}(x_1)+\frac{h^2}{2}f^{''}(x_1)+\frac{h^3}{6}f^{(3)}(x_1)+\frac{1}{6}\int_{x_1}^{x_1+h}f^{(4)}(t)(x_1+h-t)^3dt$$

$$f(x_1-h)=f(x_1)-hf^{'}(x_1)+\frac{h^2}{2}f^{''}(x_1)-\frac{h^3}{6}f^{(3)}(x_1)+\frac{1}{6}\int_{x_1}^{x_1-h}f^{(4)}(t)(x_1-h-t)^3dt$$

We have

$$f^{''}(x_1)=\frac{1}{h^2}[f(x_1-h)-2f(x_1)+f(x_1+h)]-\frac{1}{6h^2}[\int_{x_1}^{x_1+h}f^{(4)}(t)(x_1+h-t)^3dt-\int_{x_1-h}^{x_1}f^{(4)}(t)(x_1-h-t)^3dt\ ]\ (*)$$

Furthermore,

$$f(x)=f(x_1)+f^{'}(x_1)(x-x_1)+\frac{f^{''}(x_1)}{2}(x-x_1)^2+\frac{f^{(3)}(x_1)}{6}(x-x_1)^3+\frac{1}{6}\int_{x_1}^{x}f^{(4)}(t)(x-t)^3dt$$

$$\begin{aligned}\int_{x_0}^{x_2}f(x)dx&=\left[f(x_1)(x-x_1)+\frac{f^{'}(x_1)}{2}(x-x_1)^2+\frac{f^{''}(x_1)}{6}(x-x_1)^3+\frac{f^{(3)}(x_1)}{24}(x-x_1)^4\right]\Bigg|_{x_0}^{x_2}\\&\quad+\frac{1}{6}\int_{x_0}^{x_2}\int_{x_1}^{x}f^{(4)}(t)(x-t)^3dtdx\\&=2hf(x_1)+\frac{h^3}{3}f^{''}(x_1)+\frac{1}{6}\int_{x_0}^{x_2}\int_{x_1}^{x}f^{(4)}(t)(x-t)^3dtdx\\&=2hf(x_1)+\frac{h^3}{3}[f(x_0)-2f(x_1)+f(x_2)]+\textbf{error term}\quad(\text{By }(*))\\&=\frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]+\textbf{error term}\end{aligned}$$

where the **error term** is

$$-\frac{h}{18}[\int_{x_1}^{x_2}f^{(4)}(t)(x_2-t)^3dt-\int_{x_0}^{x_1}f^{(4)}(t)(x_0-t)^3dt\ ]+\frac{1}{6}\int_{x_0}^{x_2}\int_{x_1}^{x}f^{(4)}(t)(x-t)^3dtdx$$

and

$$\begin{aligned}\frac{1}{6}\int_{x_0}^{x_2}\int_{x_1}^{x}f^{(4)}(t)(x-t)^3dtdx&=\frac{1}{6}\int_{x_0}^{x_1}\int_{x_1}^{x}f^{(4)}(t)(x-t)^3dtdx+\frac{1}{6}\int_{x_1}^{x_2}\int_{x_1}^{x}f^{(4)}(t)(x-t)^3dtdx\\&=-\frac{1}{6}\int_{x_0}^{x_1}\int_{x_0}^{t}f^{(4)}(t)(x-t)^3dxdt+\frac{1}{6}\int_{x_1}^{x_2}\int_{t}^{x_2}f^{(4)}(t)(x-t)^3dxdt\\&=\frac{1}{24}\int_{x_0}^{x_1}f^{(4)}(t)(x_0-t)^4dt+\frac{1}{24}\int_{x_1}^{x_2}f^{(4)}(t)(x_2-t)^4dt\end{aligned}$$

So, the **error term** becomes

$$\int_{x_1}^{x_2} f^{(4)}(t)[-\frac{h}{18}(x_2-t)^3+\frac{1}{24}(x_2-t)^4]dt+\int_{x_0}^{x_1} f^{(4)}(t)[\frac{h}{18}(x_0-t)^3+\frac{1}{24}(x_0-t)^4]dt$$
$$=\int_0^h f^{(4)}(x_2-u)[-\frac{h}{18}u^3+\frac{1}{24}u^4]du+\int_0^h f^{(4)}(x_0+u)[-\frac{h}{18}u^3+\frac{1}{24}u^4]du$$
$$=\int_0^h [f^{(4)}(u+x_0)+f^{(4)}(x_2-u)][-\frac{h}{18}u^3+\frac{1}{24}u^4]du$$

Since $[-\frac{h}{18}u^3+\frac{1}{24}u^4]=\frac{1}{24}u^3[u-\frac{4}{3}h]<0$ for $u\in[0,h]$, by I.V.T and M.V.T,

the above intergral equals to

$$\begin{aligned}\int_0^h 2f^{(4)}(\xi(u))\ \frac{1}{24}u^3[u-\frac{4}{3}h]du &= f^{(4)}(\xi)\int_0^h \frac{1}{12}[u^4-\frac{4}{3}u^3h]du \\ &= \frac{f^{(4)}(\xi)}{12}[\frac{h^5}{5}-\frac{4}{3}\ \frac{h^4}{4}\ h] \\ &= -\frac{f^{(4)}(\xi)}{90}h^5 \qquad \square\end{aligned}$$