Brief Solutions to Quiz 01

10 pts each, total 50 pts. Extra 4 pts in problem 1.

1. How many bits does it take to store a binary floating point number of the form $\pm 1.a_1a_2\cdots a_s \times 2^e$ with $s = 11$, $a_j \in \{0,1\}$, $-6 \le e \le 7$? Write down the binary floating number representation (binary machine number, a finite sequence of 0, 1) of -0.6875.

   Ans:

   There are total 14 different exponents ($-6 \le e \le 7$). It takes 4 bits to give 14 or more different exponents ($2^4 = 16$). Total bits $= 1 + 11 + 4 = 16$ (6pts).

   $-0.6875 = -(2^{-1} + 2^{-3} + 2^{-4}) = -(0.1011)_2$ (4 pts).

   The range of the 4-bit binary exponent $c = (b_1b_2b_3b_4)_2$, $b_i = 0, 1$, is $0 \le c \le 15$. In order to cover the range $-6 \le e \le 7$, one should take $e = c - 7$, so that $e = -7$ and $e = 8$ can be reserved for underflow and overflow, respectively. With $e = c - 7$, the binary machine number is given by (extra 4 pts):

   $$-0.6875 = -(1.011)_2 \times 2^{-1} = 1 \;\; 0110 \;\; 01100000000$$

2. <u>Derive</u> an upper bound for relative error cause by chopping for the floating point system in problem 1 (also known as machine epsilon).

   Ans:

   Answer $= 2^{-11}$. One can easily modify the derivation for the decimal version given in the textbook. See also the lecture notes for detailed derivation of the binary version.

3. Solve for $x^2 - 2100x + 1 = 0$ to 15 correct digits. Explain how you find your answer (direct evaluation using 'calculator' will receive no credits).

   Ans:

   $x_1 = (2100+\sqrt{2100^2 - 4})/2 = 2.09999952380941\times 10^3$, $x_2 = 1/x_1 = 4.76190584170224\times 10^{-4}$.

4. Consider the following recursive equation $p_0 = 1, \quad p_1 = 1/3, \quad p_n = \frac{10}{3}p_{n-1} - p_{n-2}$. What is the exact solution? Is it stable? Explain.

   Ans:

   See page 34-35 of the textbook (5 pts for each question). Key points in grading the unstable part: (1) Two characteristic values with $|\lambda_2| > 1$. (2) Due to floating point truncation, the true coefficient c_2 in $c_2\lambda_2^n$ is actually non-zero.

5. How many additions/subtractions and how many multiplications/divisions does it take to evaluate $\dfrac{1}{\sum_{i=0}^{9}\frac{5^i}{i!}}$? Explain. Try to give the most efficient way of evaluation. Then perform 4-digit rounding using the subroutines given on the course homepage (or your own subroutine, if you prefer). If you don't have the most efficient way of evaluating the sum, implement your own method for partial credits.

Hint: the answer is close to, but may not equal to 7.090×10^{-3}.

Ans:

The nested sum

$$1/(((((((((x/9+1)*x/8+1)*x/7+1)*x/6+1)*x/5+1)*x/4+1)*x/3+1)*x/2+1)*x+1)$$

is most efficient (and can be arranged in a for loop). Total 9 additions/subtractions and 17 multiplications/divisions.

Analysis: 2 pts. Code: 8 pts. Non-nested version: half credit each.