# Brief Solutions to Midterm 01

Total: 116 pts. 20-29:2, 30-39:6, 40-49:11, 50-59:13, 60-69:8, 70-79:1, 80-89:2, 90-99:0, 100-:1.

1. (20 pts) Use any method to find a solution of $(1+x)^{1/3} - (1-x)^{1/3} = 10^{-10}$ to 10 correct digits.

   **Answer** : $1.500000000 \times 10^{-10}$ or $1.499999999 \times 10^{-10}$
   **Solution** : It is easy to see (by Taylor expansion) that $x = O(10^{-10})$. Therefore direct evaluation of of $(1+x)^{1/3} - (1-x)^{1/3}$ is a textbook example of losing accuracy (about 6-7 digits remaining correct). The key to 10 correct digits is the following identity

   $$a - b = (a^3 - b^3)/(a^2 + ab + b^2)$$

   that avoids the subtraction of two nearly identical numbers and gives:

   $$f(x) = \frac{2x}{(1+x)^{2/3} + (1-x^2)^{1/3} + (1-x)^{2/3}} - 10^{-10}$$

   One can get about 15 correct digits by solving $f(x) = 0$ with any numerical method (Extra 1 pts with C language).

   Without the transformation, one can only get 6-7 correct digits in the end (and get total 14 pts maximum). Some people got more than 6-7 digits for certain $x_0$, but that was pure luck. A slight perturbation of $x_0$ will result to the normal 6-7 correct digits.

2. (12 pts) Find $p_{20}$ to 10 correct digits if $p_0 = 1$, $p_1 = 0.9$, $p_n = 4p_{n-1} - 2.79p_{n-2}$.

   **Answer** : $p_{20} = 0.1215766545$ or $0.1215766546$
   **Solution** : The recursive formula is unstable since one of the characteristic values is larger than 1 in absolute value. Direct evaluation using the recursive formula gets about 6 correct digits (4 pts maximum).

   The proper solution: Solve the recursive equation to find $p_n = (0.9)^n$ (10 pts) and then evaluate $p_{20} = (0.9)^{20}$ directly (2 pts).

3. (10+6 pts)

   (a) Give a locally convergent fixed point iteration for solving $f(x) = x - 3\sin(x) - 0.01 = 0$. Give the formula and find a solution to 10 correct digits.

   (b) Give an upper bound for the number of steps it takes to reach $|x_n - x^*| < 10^{-6}$ with $x_0 = 1$.

   (a) **Answer** : There are three solutions $x^* = -2.275469641$ (or $-2.275469642$), $-0.005000031250$ (or $-0.005000031251$), $2.282246811$. Either one will get full credit.
   **Solution** :

Direct fixed point iteration with $g(x) = g_0(x) = 3\sin x + 0.01$ does not converge. Instead, a proper choice of $\beta$ and $g(x) = \beta x + (1-\beta)g_0(x)$ will result in local convergence (3 pts). To find a correct $\beta$, one could plot $(x, f(x))$ to find the approximate locations of $x^*$'s. Or Taylor expansion to find the approximate location of one of them and then get an approximate value of $g'(x^*)$. Choose $\beta$ so that $g'(x^*) \approx 0$. Some $\beta$'s that actually work include and get $\beta = 2/3, 3/2, 3/4, 4/3, \cdots$ (4 pts). Getting a correct $x^*$ gets remaining (3 pts). Giving correct interval of convergence gets (extra 2 pts). Extra 1 pts with C language.

(b) **Solution** : Any reasonable estimate (need not be optimal) for your working method in (a) gets full 6 pts.

4. (12 pts) Derive Aitken's $\Delta^2$ method from the assumption

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}$$

**Solution** :

See derivation of (2.14) on page 87 of the textbook. Partial credits for partial results.

5. (12+12 pts)

(a) Give a cubically convergent method to solve for $e^x - 1 = 0$. Give the formula and prove that it is cubically convergent (locally). If you cannot do it, do the same for a quadratically convergent method for partial credit.

(b) Find $\alpha$ and $\lambda$ (the constants in the definition of order of convergence) analytically for your method if it is applied to solve the equation $e^x - x - 1 = 0$ instead. If you cannot do it, do the same for a quadratically convergent method for partial credit (no matter whether you did cubically or quadratically convergent method in (a) ).

**Answer**:

(a)

[Cubic]

One solution is given by (there may be others)

$$x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)}\left[\frac{f(x)}{f'(x)}\right]^2. \quad \text{(4 pts)}$$

Check that $g'(p) = g''(p) = 0$ and $g^{(3)}(p) \neq 0$, and then compute

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} = \frac{|g^{(3)}(p)|}{3!}. \quad \text{(8 pts)}$$

[Quadratic]

$$x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)}. \quad \text{(2 pts)}$$

Check that $g'(p) = 0$ and $g''(p) \neq 0$, and then compute

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(p)|}{2!}. \quad \text{(4 pts)}$$

(b)

[Cubic]

For the $g(x)$ given in (a), one checks that (8 pts)

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|$$

to get $\alpha = 1$, $\lambda = 3/8$ (4 pts).

[Quadratic]

For the $g(x)$ given in (a), one checks that (4 pts)

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|$$

to get $\alpha = 1$, $\lambda = 1/2$ (2 pts).

6. (20 pts) Find $P(0.3)$ where $P(x)$ is the Lagrange polynomial interpolating the data $(0, \sin 0), (0.25, \sin 0.25), (0.5, \sin 0.5), (0.75, \sin 0.75), (1, \sin 1)$. Give formula and find the numerical value.

   **Answer** : $P(0.3) = 0.295527184899901$.
   **Solution** : Lagrange polynomial (10 pts):

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x); \; L_{n,k}(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

   Implementation to get correct answer (10 pts). C language (extra 2 pts).

7. (12 pts) Suppose that we are to construct a piecewise polynomial interpolation $S(x)$ on the data $(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$, with additional continuity conditions for $S'$, $S''$ and $S'''$ on the interior nodes $x_1, \cdots, x_{n-1}$. If we use polynomials of the same degrees on each of the interval $[x_0, x_1], \cdots, [x_{n-1}, x_n]$, what is the minimal degree needed in each interval? How many additional end conditions are needed? Explain.

   **Answer** : (6 pts) Minimal degree = 4; need 3 more end conditions.
   **Solution** : (6 pts) There are $5n$ unknowns and $5n - 3$ conditions.