1. The Heat Equation Backward-Difference Algorithm gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i,t_j)$
1	1	0.5	0.05	0.632952	0.652037
2	1	1.0	0.05	0.895129	0.883937
3	1	1.5	0.05	0.632952	0.625037
1	2	0.5	0.1	0.566574	0.552493
2	2	1.0	0.1	0.801256	0.781344
3	2	1.5	0.1	0.566574	0.552493

2. The Heat Equation Backward-Difference Algorithm gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i,t_j)$
1	1	1/3	0.05	1.59728	1.53102
2	1	2/3	0.05	−1.59728	−1.53102
1	2	1/3	0.1	1.47300	1.35333
2	2	2/3	0.1	−1.47300	−1.35333

3. The Crank-Nicolson Algorithm gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i,t_j)$
1	1	0.5	0.05	0.628848	0.652037
2	1	1.0	0.05	0.889326	0.883937
3	1	1.5	0.05	0.628848	0.625037
1	2	0.5	0.1	0.559251	0.552493
2	2	1.0	0.1	0.790901	0.781344
3	2	1.5	0.1	0.559252	0.552493

4. The Crank-Nicolson Algorithm gives the following results.

For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	w_{ij}	$u(x_i,t_j)$
1	1	1/3	0.05	1.59728	1.53102
2	1	2/3	0.05	−1.59728	−1.53102
1	2	1/3	0.1	1.47300	1.35333
2	2	2/3	0.1	−1.47300	−1.35333

5. The Forward-Difference Algorithm gives the following results.

(a) For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	5	0.8	0.5	3.035630	0
3	5	1.2	0.5	−3.035630	0
4	5	1.6	0.5	1.876122	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	10	0.8	0.5	0	0
3	10	1.2	0.5	0	0
4	10	1.6	0.5	0	0

(b) For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
3	10	0.94247780	0.5	0.4864823	0.4906936
6	10	1.88495559	0.5	0.5728943	0.5768449
9	10	2.82743339	0.5	0.1858197	0.1874283

6. The Forward-Difference Algorithm gives the following results.

(a) For $h = 0.2$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
4	10	0.8	0.4	1.166149	1.169362
8	10	1.6	0.4	1.252413	1.254556
12	10	2.4	0.4	0.4681813	0.4665473
16	10	3.2	0.4	−0.1027637	−0.1056622

(b) For $h = 0.1$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
3	10	0.3	0.4	0.5397009	0.5423003
6	10	0.6	0.4	0.6344565	0.6375122
9	10	0.9	0.4	0.2061474	0.2071403

7. The Backward-Difference Algorithm gives:

(a) For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
2	5	0.8	0.5	−0.00258	0
3	5	1.2	0.5	0.00258	0
4	5	1.6	0.5	−0.00159	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
2	10	0.8	0.5	-4.93×10^{-4}	0
3	10	1.2	0.5	4.93×10^{-4}	0
4	10	1.6	0.5	-3.05×10^{-4}	0

(b) For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
3	10	0.94247780	0.5	0.4986092	0.4906936
6	10	1.88495559	0.5	0.5861503	0.5768449
9	10	2.82743339	0.5	0.1904518	0.1874283

8. (a) For $h = 0.2$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
4	10	0.8	0.4	1.176752	1.169362
8	10	1.6	0.4	1.259495	1.254556
12	10	2.4	0.4	0.4628134	0.4665473
16	10	3.2	0.4	−0.1123064	−0.1056622

(b) For $h = 0.1$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
3	10	0.3	0.4	0.5482691	0.5423003
6	10	0.6	0.4	0.6445290	0.6375123
9	10	0.9	0.4	0.2094202	0.2071403

9. The Crank-Nicolson Algorithm gives the following results.

(a) For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	5	0.8	0.5	8.2×10^{-7}	0
3	5	1.2	0.5	-8.2×10^{-7}	0
4	5	1.6	0.5	5.1×10^{-7}	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	10	0.8	0.5	-2.6×10^{-6}	0
3	10	1.2	0.5	2.6×10^{-6}	0
4	10	1.6	0.5	-1.6×10^{-6}	0

(b) For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
3	10	0.94247780	0.5	0.4926589	0.4906936
6	10	1.88495559	0.5	0.5791553	0.5768449
9	10	2.82743339	0.5	0.1881790	0.1874283

10. The Crank-Nicolson Algorithm gives the following results.

(a) For $h = 0.2$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
4	10	0.8	0.4	1.171532	1.169362
8	10	1.6	0.4	1.256005	1.254556
12	10	2.4	0.4	0.4654499	0.4665473
16	10	3.2	0.4	−0.1076139	−0.1056622

(b) For $h = 0.1$ and $k = 0.04$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
3	10	0.3	0.4	0.5440532	0.5423003
6	10	0.6	0.4	0.6395728	0.6375122
9	10	0.9	0.4	0.2078098	0.2071403

11. Using Richardson's method gives:

(a) Using $h = 0.4$ and $k = 0.1$ leads to meaningless results. Using $h = 0.4$ and $k = 0.05$ again gives meaningless answers. Letting $h = 0.4$ and $k = 0.005$ produces the following:

i	j	x_i	t_j	w_{ij}
1	100	0.4	0.5	−165.405
2	100	0.8	0.5	267.613
3	100	1.2	0.5	−267.613
4	100	1.6	0.5	165.405

The instability of Richardson's method gives very poor results.

(b)

i	j	x_i	t_j	$w(x_{ij})$
3	10	0.9424778	0.5	0.46783396
6	10	1.8849556	0.5	0.54995267
9	10	2.8274334	0.5	0.17871220

12. Using Richardson's method gives:

(a) For $h = 0.2$ and $k = 0.04$

i	j	x_i	t_j	$w(x_{ij})$
4	10	0.8	0.4	1.1406275
8	10	1.6	0.4	1.2315952
12	10	2.4	0.4	0.47267557
16	10	3.2	0.4	−0.08733023

(b) For $h = 0.1$ and $k = 0.04$

i	j	x_i	t_j	$w(x_{ij})$
2	10	0.2	0.4	0.37945980
4	10	0.4	0.4	0.61397885
6	10	0.6	0.4	0.61397885
8	10	0.8	0.4	0.37945980

13. We have

$$a_{11}v_1^{(i)} + a_{12}v_2^{(i)} = (1-2\lambda)\sin\frac{i\pi}{m} + \lambda\sin\frac{2\pi i}{m}$$

and

$$\begin{aligned}\mu_i v_1^{(i)} &= \left[1 - 4\lambda\left(\sin\frac{i\pi}{2m}\right)^2\right]\sin\frac{i\pi}{m} = \left[1 - 4\lambda\left(\sin\frac{i\pi}{2m}\right)^2\right]\left(2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m}\right)\\ &= 2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m} - 8\lambda\left(\sin\frac{i\pi}{2m}\right)^3\cos\frac{i\pi}{2m}.\end{aligned}$$

However,

$$\begin{aligned}(1-2\lambda)\sin\frac{i\pi}{m} + \lambda\sin\frac{2\pi i}{m} &= 2(1-2\lambda)\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m} + 2\lambda\sin\frac{i\pi}{m}\cos\frac{i\pi}{m}\\ &= 2(1-2\lambda)\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m}\\ &\quad + 2\lambda\left[2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m}\right]\left[1 - 2\left(\sin\frac{i\pi}{2m}\right)^2\right]\\ &= 2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m} - 8\lambda\cos\frac{i\pi}{2m}\left[\sin\frac{i\pi}{2m}\right]^3.\end{aligned}$$

Thus

$$a_{11}v_1^{(i)} + a_{12}v_2^{(i)} = \mu_i v_1^{(i)}.$$

Further,

$$\begin{aligned}a_{j,j-1}v_{j-1}^{(i)} + a_{j,j}v_j^{(i)} + a_{j,j+1}v_{j+1}^{(i)} &= \lambda\sin\frac{i(j-1)\pi}{m} + (1-2\lambda)\sin\frac{ij\pi}{m} + \lambda\sin\frac{i(j+1)\pi}{m}\\ &= \lambda\left(\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m} - \sin\frac{i\pi}{m}\cos\frac{ij\pi}{m}\right) + (1-2\lambda)\sin\frac{ij\pi}{m}\\ &\quad + \lambda\left(\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m} + \sin\frac{i\pi}{m}\cos\frac{ij\pi}{m}\right)\\ &= \sin\frac{ij\pi}{m} - 2\lambda\sin\frac{ij\pi}{m} + 2\lambda\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m}\\ &= \sin\frac{ij\pi}{m} + 2\lambda\sin\frac{ij\pi}{m}\left(\cos\frac{i\pi}{m} - 1\right)\end{aligned}$$

and

$$\begin{aligned}\mu_i v_j^{(i)} &= \left[1 - 4\lambda\left(\sin\frac{i\pi}{2m}\right)^2\right]\sin\frac{ij\pi}{m} = \left[1 - 4\lambda\left(\frac{1}{2} - \frac{1}{2}\cos\frac{i\pi}{m}\right)\right]\sin\frac{ij\pi}{m}\\ &= \left[1 + 2\lambda\left(\cos\frac{i\pi}{m} - 1\right)\right]\sin\frac{ij\pi}{m},\end{aligned}$$

so

$$a_{j,j-1}v_{j-1}^{(i)} + a_{j,j}v_j^{(i)} + a_{j,j+1}v_j^{(i)} = \mu_i v_j^{(i)}.$$

Similarly,

$$a_{m-2,m-1}v_{m-2}^{(i)} + a_{m-1,m-1}v_{m-1}^{(i)} = \mu_i v_{m-1}^{(i)},$$

so $A\mathbf{v}^{(i)} = \mu_i\mathbf{v}^{(i)}$.

14. We have

$$a_{11}v_1^{(i)} + a_{12}v_2^{(i)} = (1+2\lambda)\sin\frac{i\pi}{m} - \lambda\sin\frac{2\pi i}{m} = (1+2\lambda)\sin\frac{i\pi}{m} - 2\lambda\sin\frac{i\pi}{m}\cos\frac{i\pi}{m}$$
$$= \sin\frac{i\pi}{m}\left[1+2\lambda\left(1-\cos\frac{i\pi}{m}\right)\right]$$

and

$$\mu_i v_1^{(i)} = \left[1+4\lambda\left(\sin\frac{i\pi}{2m}\right)^2\right]\sin\frac{i\pi}{m} = \left[1+2\lambda\left(1-\cos\frac{i\pi}{m}\right)\right]\sin\frac{i\pi}{m} = a_{11}v_1^{(i)} + a_{12}v_2^{(i)}.$$

In general,

$$\begin{aligned} a_{j,j-1}v_{j-1}^{(i)} + a_{j,j}v_j^{(i)} + a_{j,j+1}v_{j+1}^{(i)} &= -\lambda\sin\frac{i(j-1)\pi}{m} + (1+2\lambda)\sin\frac{ij\pi}{m} - \lambda\sin\frac{i(j+1)\pi}{m} \\ &= -\lambda\left(\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m} - \sin\frac{i\pi}{m}\cos\frac{ij\pi}{m}\right) + (1+2\lambda)\sin\frac{ij\pi}{m} \\ &\quad -\lambda\left(\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m} + \sin\frac{i\pi}{m}\cos\frac{ij\pi}{m}\right) \\ &= -2\lambda\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m} + (1+2\lambda)\sin\frac{ij\pi}{m} \\ &= \left[1+2\lambda\left(1-\cos\frac{i\pi}{m}\right)\right]\sin\frac{ij\pi}{m} = \mu_i v_j^{(i)}. \end{aligned}$$

Similarly,

$$a_{m-2,m-1}v_{m-2}^{(i)} + a_{m-1,m-1}v_{m-1}^{(i)} = \mu_i v_{m-1}^{(i)}.$$

Thus, $A\mathbf{v}^{(i)} = \mu_i\mathbf{v}^{(i)}$. Since A is symmetric with positive eigenvalues, A is positive definite. Further,

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| = 2\lambda < 1+2\lambda = |a_{ii}|, \quad \text{for } 1 \le i \le n,$$

so A is diagonally dominant.

15. To modify Algorithm 12.2, change the following:

Step 7 Set

$t = jk;$
$w_0 = \phi(t);$
$z_1 = (w_1 + \lambda w_0)/l_1.$
$w_m = \psi(t).$

Step 8 For $i = 2, \ldots, m-2$ set

$$z_i = (w_i + \lambda z_{i-1})/l_i;$$

Set

$$z_{m-1} = (w_{m-1} + \lambda w_m + \lambda z_{m-2})/l_{m-1}.$$

Step 11 OUTPUT (t);
For $i = 0, \ldots, m$ set $x = ih$;
OUTPUT (x, w_i).

To modify Algorithm 12.3, change the following:

Step 1 Set

$h = l/m;$
$k = T/N;$
$\lambda = \alpha^2 k/h^2;$
$w_m = \psi(0);$
$w_0 = \phi(0).$

Step 7 Set

$t = jk;$
$z_1 = \left[(1-\lambda)w_1 + \frac{\lambda}{2}w_2 + \frac{\lambda}{2}{}_0 + \frac{\lambda}{2}\phi(t)\right]/l_1;$
$w_0 = \phi(t).$

Step 8 For $i = 2, \ldots, m-2$ set

$$z_i = \left[(1-\lambda)w_i + \frac{\lambda}{2}(w_{i+1} + w_{i-1} + z_{i-1})\right]/l_i;$$

Set

$$z_{m-1} = \left[(1-\lambda)w_{m-1} + \frac{\lambda}{2}(w_m + w_{m-2} + z_{m-2} + \psi(t))\right]/l_{m-1};$$
$$w_m = \psi(t).$$

Step 11 OUTPUT (t);
For $i = 0, \ldots, m$ set $x = ih$;
OUTPUT (x, w_i).

16. The modifications of Algorithms 12.2 and 12.3 give the following results:

(a) For modified Algorithm 12.2 we have

i	j	x_i	t_j	w_{ij}
3	25	0.3	0.25	0.2883460
5	25	0.5	0.25	0.3468410
8	25	0.8	0.25	0.2169217

(b) For modified Algorithm 12.3 we have

i	j	x_i	t_j	w_{ij}
3	25	0.3	0.25	0.2798737
5	25	0.5	0.25	0.3363686
8	25	0.8	0.25	0.2107662

17. To modify Algorithm 12.2, change the following:

STEP 7 Set

$t = jk;$
$w_0 = \phi(t);$
$z_1 = (w_1 + \lambda w_0)/l_1.$
$w_m = \psi(t).$

STEP 8 For $i = 2, \ldots, m-2$ set

$$z_i = (w_i + \lambda z_{i-1})/l_i;$$

Set

$$z_{m-1} = (w_{m-1} + \lambda w_m + \lambda z_{m-2})/l_{m-1}.$$

STEP 11 OUTPUT (t);
For $i = 0, \ldots, m$ set $x = ih$;
OUTPUT (x, w_i).

To modify Algorithm 12.3, change the following:

STEP 1 Set

$h = l/m;$
$k = T/N;$
$\lambda = \alpha^2 k/h^2;$
$w_m = \psi(0);$
$w_0 = \phi(0).$

STEP 7 Set

$t = jk;$
$z_1 = \left[(1-\lambda)w_1 + \frac{\lambda}{2}w_2 + \frac{\lambda}{2}{}_0 + \frac{\lambda}{2}\phi(t)\right]/l_1;$
$w_0 = \phi(t).$

STEP 8 For $i = 2, \ldots, m-2$ set

$$z_i = \left[(1-\lambda)w_i + \frac{\lambda}{2}(w_{i+1} + w_{i-1} + z_{i-1})\right]/l_i;$$

Set

$$z_{m-1} = \left[(1-\lambda)w_{m-1} + \frac{\lambda}{2}(w_m + w_{m-2} + z_{m-2} + \psi(t))\right]/l_{m-1};$$
$$w_m = \psi(t).$$

STEP 11 OUTPUT (t);
For $i = 0, \ldots, m$ set $x = ih$;
OUTPUT (x, w_i).

18. The approximations to the temperature distributions using Algorithms 12.2 and 12.3 are given in the following table:

i	j	x_i	t_j	w_{ij}(Algorithm 12.3)	w_{ij}(Algorithm 12.2)
3	10	0.3	0.225	1.223279	1.207730
6	10	0.75	0.225	1.862358	1.836564
10	10	1.35	0.225	0.701087	0.692834

19. (a) The approximate temperature at some typical points is given in the table.

i	j	r_i	t_j	$w_{i,j}$
1	20	0.6	10	137.6753
2	20	0.7	10	245.9678
3	20	0.8	10	340.2862
4	20	0.9	10	424.1537

(b) The strain is approximately $I = 1242.537$.