1. The Piecewise Linear Algorithm gives

$$\phi(x) = -0.07713274\phi_1(x) - 0.07442678\phi_2(x).$$

The actual values are

$$y(x_1) = -0.07988545 \quad \text{and} \quad y(x_2) = -0.07712903.$$

2. The Piecewise Linear Algorithm gives

$$\phi(x) = -0.2552629\phi_1(x) - 0.1633565\phi_2(x).$$

The actual values are

$$y(x_1) = -0.24 \quad \text{and} \quad y(x_2) = -0.16.$$

3. The Piecewise Linear Algorithm gives the results in the following tables.

(a)

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	−0.212333	−0.21
6	0.6	−0.241333	−0.24
9	0.9	−0.090333	−0.09

(b)

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	0.1815138	0.1814273
6	0.6	0.1805502	0.1804753
9	0.9	0.05936468	0.05934303

(c)

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.3585989	−0.3585641
10	0.50	−0.5348383	−0.5347803
15	0.75	−0.4510165	−0.4509614

(d)

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.1846134	−0.1845204
10	0.50	−0.2737099	−0.2735857
15	0.75	−0.2285169	−0.2284204

4. The Cubic Spline Algorithm gives the results in the following tables.

(a)

i	x_i	$\phi(x_i)$	y_i
1	0.25	−0.0712415	−0.0712308
2	0.5	−0.0944237	−0.0944091
3	0.75	−0.0681742	−0.0681651

(b)

i	x_i	$\phi(x_i)$	y_i
1	0.25	−0.1875	−0.1875
2	0.5	−0.25	−0.25
3	0.75	−0.1875	−0.1875

5. The Cubic Spline Algorithm gives the results in the following tables.

(a)

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	−0.2100000	−0.21
6	0.6	−0.2400000	−0.24
9	0.9	−0.0900000	−0.09

(b)

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	0.1814269	0.1814273
6	0.6	0.1804753	0.1804754
9	0.9	0.05934321	0.05934303

(c)

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.3585639	−0.3585641
10	0.50	−0.5347779	−0.5347803
15	0.75	−0.4509109	−0.4509614

(d)

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.1845191	−0.1845204
10	0.50	−0.2735833	−0.2735857
15	0.75	−0.2284186	−0.2284204

6. With $z(x) = y(x) - \beta x - \alpha(1-x)$, we have

$$z(0) = y(0) - \alpha = \alpha - \alpha = 0 \quad \text{and} \quad z(1) = y(1) - \beta = \beta - \beta = 0.$$

Further, $z'(x) = y'(x) - \beta + \alpha$. Thus

$$y(x) = z(x) + \beta x + \alpha(1-x) \quad \text{and} \quad y'(x) = z'(x) + \beta - \alpha.$$

Substituting for y and y' in the differential equation gives

$$-\frac{d}{dx}(p(x)z' + p(x)(\beta - \alpha)) + q(x)(z + \beta x + \alpha(1-x)) = f(x).$$

Simplifying gives the differential equation

$$-\frac{d}{dx}(p(x)z') + q(x)z = f(x) + (\beta - \alpha)p'(x) - [\beta x + \alpha(1-x)]q(x).$$

7. Exercise 6 and the Piecewise Linear Algorithm give:

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	1.0408182	1.0408182
6	0.6	1.1065307	1.1065306
9	0.9	1.3065697	1.3065697

8. The Cubic Spline Algorithm gives the results in the following table.

x_i	$\phi_i(x)$	$y(x_i)$
0.3	1.0408183	1.0408182
0.5	1.1065307	1.1065301
0.9	1.3065697	1.3065697

9. A change in variable $w = (x-a)/(b-a)$ gives the boundary value problem

$$-\frac{d}{dw}(p((b-a)w+a)y') + (b-a)^2 q((b-a)w+a)y = (b-a)^2 f((b-a)w+a),$$

where $0 < w < 1$, $y(0) = \alpha$, and $y(1) = \beta$. Then Exercise 6 can be used.

10. If $\sum_{i=1}^{n} c_i\phi_i(x) = 0$, for $0 \leq x \leq 1$, then for any j, we have $\sum_{i=1}^{n} c_i\phi_i(x_j) = 0$.

But

$$\phi_i(x_j) = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$

so $c_j\phi_j(x_j) = c_j = 0$. Hence the functions are linearly independent.

11. Suppose $\phi(x) = \sum_{i=0}^{n+1} c_i\phi_i(x) = 0$, for all x in $[0, 1]$. At the nodes $x_i, i = 0, \ldots, n+1$, we have

$$\phi_0(x_i) = \begin{cases} 1/4, & \text{if } i = 1 \\ 0, & \text{otherwise;} \end{cases}$$

$$\phi_1(x_i) = \begin{cases} 1, & \text{if } i = 1 \\ 1/4, & \text{if } i = 2 \\ 0, & \text{otherwise;} \end{cases}$$

$$\phi_n(x_i) = \begin{cases} 1, & \text{if } i = n \\ 1/4, & \text{if } i = n-1 \\ 0, & \text{otherwise;} \end{cases}$$

$$\phi_{n+1}(x_i) = \begin{cases} 1/4, & \text{if } i = n \\ 0, & \text{otherwise;} \end{cases}$$

and for $j = 2, 3, \ldots, n-1$,

$$\phi_j(x_i) = \begin{cases} 1, & \text{if } i = j \\ 1/4, & \text{if } i = j-1 \text{ or } i = j+1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} 0 =& \phi(x_1) = \frac{1}{4}c_0 + c_1 + \frac{1}{4}c_2 \\ 0 =& \phi(x_2) = \frac{1}{4}c_1 + c_2 + \frac{1}{4}c_3 \\ & \vdots \\ 0 =& \phi(x_{n-1}) = \frac{1}{4}c_{n-2} + c_{n-1} + \frac{1}{4}c_n \\ 0 =& \phi(x_n) = \frac{1}{4}c_{n-1} + c_n + \frac{1}{4}c_{n+1}. \end{aligned}$$

Since $\phi'(0) = \phi'(1) = 0$, we have

$$0 = \frac{3}{h}c_0 + \frac{1.5}{h}c_1, \quad \text{so} \quad 0 = 3c_0 + 1.5c_1$$

and

$$0 = -\frac{1.5}{h}c_n - \frac{3}{h}c_{n+1}, \quad \text{so} \quad 0 = 1.5c_n + 3c_{n+1}.$$

Thus,

$$\begin{bmatrix} 3 & 1.5 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0.25 & 1 & 0.25 & \ddots & & & & \vdots \\ 0 & 0.25 & 1 & 0.25 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0.25 & 1 & 0.25 & 0 \\ \vdots & & & & \ddots & 0.25 & 1 & 0.25 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1.5 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which can be written as the linear system $A\mathbf{c} = \mathbf{0}$. The matrix A is strictly diagonally dominant and, hence, nonsingular. So the only solution to the linear system is $\mathbf{c} = \mathbf{0}$, and $\{\phi_0, \phi_1, \dots, \phi_n, \phi_{n+1}\}$ is linearly independent.

12. Let $\mathbf{c} = (c_1, \dots, c_n)^t$ be any vector and let $\phi(x) = \sum_{j=1}^{n} c_j\phi_j(x)$. Then

$$\begin{aligned}
\mathbf{c}^t A\mathbf{c} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}c_ic_j = \sum_{i=1}^{n}\sum_{j=i-1}^{i+1} a_{ij}c_ic_j \\
&= \sum_{i=1}^{n}\left[\int_0^1 \{p(x)c_i\phi_i'(x)c_{i-1}\phi_{i-1}'(x) + q(x)c_i\phi_i(x)c_{i-1}\phi_{i-1}(x)\}\,dx\right. \\
&\quad + \int_0^1 \{p(x)c_i^2[\phi_i'(x)]^2 + q(x)c_i^2[\phi_i'(x)]^2\}\,dx \\
&\quad \left. + \int_0^1 \{p(x)c_i\phi_i'(x)c_{i+1}\phi_{i+1}'(x) + q(x)c_i\phi_i(x)c_{i+1}\phi_{i+1}(x)\}\,dx\right] \\
&= \int_0^1 \{p(x)[\phi'(x)]^2 + q(x)[\phi(x)]^2\}\,dx.
\end{aligned}$$

So $\mathbf{c}^t A\mathbf{c} \geq 0$ with equality only if $\mathbf{c} = \mathbf{0}$. Since A is also symmetric, A is positive definite.

13. For $\mathbf{c} = (c_0, c_1, \ldots, c_{n+1})^t$ and $\phi(x) = \sum_{i=0}^{n+1} c_i\phi_i(x)$, we have

$$\mathbf{c}^t A\mathbf{c} = \int_0^1 p(x)[\phi'(x)]^2 + q(x)[\phi(x)]^2\,dx.$$

But $p(x) > 0$ and $q(x)[\phi(x)]^2 \geq 0$, so $\mathbf{c}^t A\mathbf{c} \geq 0$, and it can be 0, for $\mathbf{x} \neq 0$, only if $\phi'(x) \equiv 0$ on $[0, 1]$. However, $\{\phi_0', \phi_1', \ldots, \phi_{n+1}'\}$ is linearly independent, so $\phi'(x) \neq 0$ on $[0, 1]$ and $\mathbf{c}^t A\mathbf{c} = 0$ if and only if $\mathbf{c} = \mathbf{0}$.