1. The Linear Finite-Difference Algorithm gives following results.

(a)

| $i$ | $x_i$ | $w_{1i}$ | $y(x_i)$ |
|---|---|---|---|
| 1 | 0.5 | 0.83333333 | 0.82402714 |

(b)

| $i$ | $x_i$ | $w_{1i}$ | $y(x_i)$ |
|---|---|---|---|
| 1 | 0.25 | 0.39512472 | 0.39367669 |
| 2 | 0.5 | 0.82653061 | 0.82402714 |
| 3 | 0.75 | 1.33956916 | 1.33708613 |

(c) Extrapolation gives

$$y\left(\frac{1}{2}\right) \approx \frac{4(0.82653061) - 0.83333333}{3} = 0.82426304$$

2. The Linear Finite-Difference Algorithm gives following results.

(a)

| $i$ | $x_i$ | $w_{1i}$ | $y(x_i)$ |
|---|---|---|---|
| 1 | $\pi/4$ | $-0.28287080$ | $-0.282842712$ |

(b)

| $i$ | $x_i$ | $w_{1i}$ | $y(x_i)$ |
|---|---|---|---|
| 1 | $\pi/8$ | $-0.31568540$ | $-0.31543220$ |
| 2 | $\pi/4$ | $-0.28290585$ | $-0.282842712$ |
| 3 | $3\pi/8$ | $-0.20699563$ | $-0.20719298$ |

(c) Extrapolation gives

$$y\left(\frac{\pi}{4}\right) \approx \frac{4(-0.28290585) - (-0.28287080)}{3} = -0.282917533.$$

3. The Linear Finite-Difference Algorithm gives the results in the following tables.

(a)

| $i$ | $x_i$ | $w_i$ | $y(x_i)$ |
|---|---|---|---|
| 2 | 0.2 | 1.018096 | 1.0221404 |
| 5 | 0.5 | 0.5942743 | 0.59713617 |
| 7 | 0.7 | 0.6514520 | 0.65290384 |

(b)

| $i$ | $x_i$ | $w_i$ | $y(x_i)$ |
|---|---|---|---|
| 5 | 1.25 | 0.16797186 | 0.16762427 |
| 10 | 1.50 | 0.45842388 | 0.45819349 |
| 15 | 1.75 | 0.60787334 | 0.60777401 |

(c)

| $i$ | $x_i$ | $w_{1i}$ | $y(x_i)$ |
|---|---|---|---|
| 3 | 0.3 | −0.5183084 | −0.5185728 |
| 6 | 0.6 | −0.2192657 | −0.2195247 |
| 9 | 0.9 | −0.0405748 | −0.04065697 |

(d)

| $i$ | $x_i$ | $w_{1i}$ | $y(x_i)$ |
|---|---|---|---|
| 3 | 1.3 | 0.0654387 | 0.0655342 |
| 6 | 1.6 | 0.0773936 | 0.0774595 |
| 9 | 1.9 | 0.0305465 | 0.0305621 |

4. The Linear Finite-Difference Algorithm gives the results in the following tables.

(a)

| $i$ | $x_i$ | $w_{1i}$ | $y(x_i)$ |
|---|---|---|---|
| 1 | 0.15707963 | 1.05260081 | 1.05248562 |
| 2 | 0.31415927 | 1.07922974 | 1.07905555 |
| 3 | 0.47123890 | 1.07922974 | 1.07905555 |
| 4 | 0.62831853 | 1.05260081 | 1.05248562 |

(b)

| $i$ | $x_i$ | $w_i$ | $y(x_i)$ |
|---|---|---|---|
| 1 | 0.15707963 | −0.06141845 | −0.06062540 |
| 2 | 0.31415927 | −0.09240491 | −0.09119581 |
| 3 | 0.47123890 | −0.09080499 | −0.08961338 |
| 4 | 0.62831853 | −0.05825827 | −0.05749950 |

(c)

| $i$ | $x_i$ | $w_i$ | $y(x_i)$ |
|---|---|---|---|
| 5 | 1.25 | 0.64328225 | 0.64314355 |
| 10 | 1.50 | 0.68332838 | 0.68324289 |
| 15 | 1.75 | 0.69230217 | 0.69226885 |

(d)

| $i$ | $x_i$ | $w_i$ | $y(x_i)$ |
|---|---|---|---|
| 3 | 0.6 | −0.70664241 | −0.71228492 |
| 5 | 1.0 | −1.63674050 | −1.64085909 |
| 8 | 1.6 | −3.52936107 | −3.52075148 |

5. The Linear Finite-Difference Algorithm gives the results in the following tables.

| $i$ | $x_i$ | $w_i(h=0.1)$ | $i$ | $x_i$ | $w_i(h=0.05)$ |
|---|---|---|---|---|---|
| 3 | 0.3 | 0.05572807 | 6 | 0.3 | 0.05132396 |
| 6 | 0.6 | 0.00310518 | 12 | 0.6 | 0.00263406 |
| 9 | 0.9 | 0.00016516 | 18 | 0.9 | 0.00013340 |

6. The Linear Finite-Difference Algorithm with the extrapolation in Example 2 gives:

(a)

| $x_i$ | $w_i(h=0.1)$ | $w_i(h=0.05)$ | $w_i(h=0.025)$ | $\text{Ext}_{1i}$ | $\text{Ext}_{2i}$ | $\text{Ext}_{3i}$ |
|---|---|---|---|---|---|---|
| 0.2 | 1.01809654 | 1.02113909 | 1.02189067 | 1.02215327 | 1.02214120 | 1.02214039 |
| 0.4 | 0.64736665 | 0.65004438 | 0.65070691 | 0.65093696 | 0.65092775 | 0.65092714 |
| 0.6 | 0.60014996 | 0.60175137 | 0.60214815 | 0.60228517 | 0.60228041 | 0.60228009 |
| 0.8 | 0.73896130 | 0.73961176 | 0.73977312 | 0.73982858 | 0.73982691 | 0.73982680 |

(b)

| $x_i$ | $w_i(h=0.05)$ | $w_i(h=0.025)$ | $w_i(h=0.0125)$ | $\text{Ext}_{1i}$ | $\text{Ext}_{2i}$ | $\text{Ext}_{3i}$ |
|---|---|---|---|---|---|---|
| 1.2 | 0.07795820 | 0.07769625 | 0.07763091 | 0.07760893 | 0.07760913 | 0.07760914 |
| 1.4 | 0.36654278 | 0.36632776 | 0.36627411 | 0.36625609 | 0.36625623 | 0.36625624 |
| 1.6 | 0.52914512 | 0.52901406 | 0.52898134 | 0.52897037 | 0.52897043 | 0.52897044 |
| 1.8 | 0.62871452 | 0.62865682 | 0.62864241 | 0.62863759 | 0.62863761 | 0.62863761 |

7. (a) The approximate deflections are shown in the following table.

| $i$ | $x_i$ | $w_{1i}$ |
|---|---|---|
| 5 | 30 | 0.0102808 |
| 10 | 60 | 0.0144277 |
| 15 | 90 | 0.0102808 |

(b) Yes, the maximum error on the interval is within 0.2 in.

(c) Yes, the maximum deflection occurs at $x = 60$. The exact solution is within tolerance, but the approximation is not.

8. The approximate deflection at 1-in. intervals is give in the following table.

| $i$ | $x_i$ | $w_i$ |
|---|---|---|
| 10 | 10.0 | 0.1098549 |
| 20 | 20.0 | 0.1761424 |
| 25 | 25.0 | 0.1849608 |
| 30 | 30.0 | 0.1761424 |
| 40 | 40.0 | 0.1098549 |

9. First we have

$$\left|\frac{h}{2}p(x_i)\right| \leq \frac{hL}{2} < 1,$$

so

$$\left|-1 - \frac{h}{2}p(x_i)\right| = 1 + \frac{h}{2}p(x_i) \quad \text{and} \quad \left|-1 + \frac{h}{2}p(x_i)\right| = 1 - \frac{h}{2}p(x_i).$$

Therefore,

$$\left|-1 - \frac{h}{2}p(x_i)\right| + \left|-1 + \frac{h}{2}p(x_i)\right| = 2 \leq 2 + h^2 q(x_i),$$

for $2 \leq i \leq N - 1$.

Since

$$\left|-1 + \frac{h}{2}p(x_1)\right| < 2 \leq 2 + h^2 q(x_1) \quad \text{and} \quad \left|-1 - \frac{h}{2}p(x_N)\right| < 2 \leq 2 + h^2 q(x_N),$$

Theorem 6.31 implies that the linear system (11.19) has a unique solution.

10. Let $q(x) \geq w > 0$ on $[a, b]$. Then using the sixth Taylor polynomial gives

$$\frac{y(x_{i+1}) - y(x_{i-1})}{2h} = y'(x_i) + \frac{h^2}{6} y'''(x_i) + \frac{h^4}{120} y^{(5)}(x_i) + O\left(h^5\right)$$

and

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = y''(x_i) + \frac{h^2}{12} y^{(4)}(x_i) + O\left(h^4\right).$$

Thus

$$\left(2 + h^2 q(x_i)\right) y(x_i) - \left(1 - \frac{h}{2} p(x_i)\right) y(x_{i+1}) - \left(1 + \frac{h}{2} p(x_i)\right) y(x_{i-1}) + h^2 r(x_i)$$

$$= p(x_i) \frac{h^4}{6} y'''(x_i) - \frac{h^4}{12} y^{(4)}(x_i) +] O\left(h^6\right).$$

Subtracting $h^2$ times Equation (11.18) gives

$$\begin{aligned} \left(2 + h^2 q(x_i)\right)\left(y(x_i) - w_i\right) = {} & \left(1 - \frac{h}{2} p(x_i)\right)\left(y(x_{i+1}) - w_{i+1}\right) \\ & + \left(1 + \frac{h}{2} p(x_i)\right)\left(y(x_{i-1}) - w_{i-1}\right) \\ & + \left[\frac{p(x_i)}{6} y'''(x_i) - \frac{1}{12} y^{(4)}(x_i)\right] h^4 + O\left(h^6\right). \end{aligned}$$

Let $E = \max_{0 \leq i \leq N+1} |y(x_i) - w_i|$. Then since $\left|\frac{h}{2} p(x_i)\right| < 1$,

$$\left(2 + h^2 q(x_i)\right)\left(y(x_i) - w_i\right) \leq 2E + h^4 \left|\frac{p(x_i)}{6} y'''(x_i) - \frac{1}{12} y^{(4)}(x_i)\right| + O\left(h^6\right).$$

Let $K_1 = \max_{a \leq x \leq b} |y'''(x)|$ and $K_2 = \max_{a \leq x \leq b} \left|y^{(4)}(x)\right|$. If $q(x_i) \geq w$, then

$$\left(2 + h^2 w\right) E \leq 2E + h^4 \left[\frac{LK_1}{6} + \frac{K_2}{12}\right] + O\left(h^6\right)$$

and

$$E \leq h^2 \left[\frac{2LK_1 + K_2}{12w}\right] + O\left(h^4\right).$$