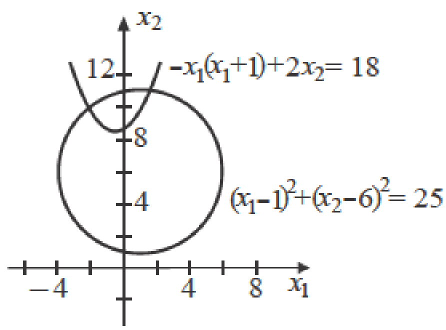
1. Use Theorem 10.5

2. One example is $\mathbf{F}(x_1, x_2) = \left(1, \dfrac{1}{(x_1-1)^2 + x_2^2}\right)^t$.

3. Use Theorem 10.5 for each of the partial derivatives.

4. The solutions are near $(-1.5, 10.5)$ and $(2, 11)$.

   (a) The graphs are shown in the figure below.



   (b) Use

$$\mathbf{G}_1(\mathbf{x}) = \left(-0.5 + \sqrt{2x_2 - 17.75}, 6 + \sqrt{25 - (x_1-1)^2}\right)^t$$

and

$$\mathbf{G}_2(\mathbf{x}) = \left(-0.5 - \sqrt{2x_2 - 17.75}, 6 + \sqrt{25 - (x_1-1)^2}\right)^t.$$

For $\mathbf{G}_1(\mathbf{x})$ with $\mathbf{x}^{(0)} = (2, 11)^t$, we have $\mathbf{x}^{(9)} = (1.5469466, 10.969994)^t$, and for $\mathbf{G}_2(\mathbf{x})$ with $\mathbf{x}^{(0)} = (-1.5, 10.5)$, we have $\mathbf{x}^{(34)} = (-2.000003, 9.999996)^t$.

5. (a) Continuity properties can be easily shown. Moreover,

$$\frac{8}{10} \le \frac{x_1^2 + x_2^2 + 8}{10} \le 1.25$$

and

$$\frac{8}{10} \le \frac{x_1 x_2^2 + x_1 + 8}{10} \le 1.2875,$$

so $\mathbf{G}(\mathbf{x}) \in D$, whenever $\mathbf{x} \in D$.
Further,

$$\frac{\partial g_1}{\partial x_1} = \frac{2x_1}{10} \quad \text{so} \quad \left|\frac{\partial g_1(\mathbf{x})}{\partial x_1}\right| \le \frac{3}{10}, \quad \frac{\partial g_1}{\partial x_2} = \frac{2x_2}{10} \quad \text{so} \quad \left|\frac{\partial g_2(\mathbf{x})}{\partial x_2}\right| \le \frac{3}{10},$$

$$\frac{\partial g_2}{\partial x_1} = \frac{x_2^2+1}{10} \quad \text{so} \quad \left|\frac{\partial g_2(\mathbf{x})}{\partial x_1}\right| \le \frac{3.25}{10}, \quad \text{and} \quad \frac{\partial g_2}{\partial x_2} = \frac{2x_1 x_2}{10} \quad \text{so} \quad \left|\frac{\partial g_2(\mathbf{x})}{\partial x_2}\right| \le \frac{4.5}{10}.$$

Since

$$\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \le 0.45 = \frac{0.9}{2},$$

for $i, j = 1, 2$, all hypothesis of Theorem 10.6 have been satisfied, and $\mathbf{G}$ has a unique fixed point in D.

(b) With $\mathbf{x}^{(0)} = (0,0)^t$ and tolerance 10^{-5}, we have $\mathbf{x}^{(13)} = (0.9999973, 0.9999973)^t$.

(c) With $\mathbf{x}^{(0)} = (0,0)^t$ and tolerance 10^{-5}, we have $\mathbf{x}^{(11)} = (0.9999984, 0.9999991)^t$.

6. (a) $\mathbf{G} = \left(x_2/\sqrt{5}, 0.25(\sin x_1 + \cos x_2)\right)^t$ and $D = \{(x_1, x_2) \mid 0 \le x_1, x_1 \le 1\}$.

(b) With $\mathbf{x}^{(0)} = \left(\frac{1}{2}, \frac{1}{2}\right)^t$, we have $\mathbf{x}^{(10)} = (0.1212440, 0.2711065)^t$.

(c) With $\mathbf{x}^{(0)} = \left(\frac{1}{2}, \frac{1}{2}\right)^t$, we have $\mathbf{x}^{(5)} = (0.1212421, 0.2711052)^t$.

7. (a) With $\mathbf{x}^{(0)} = (1,1,1)^t$, we have $\mathbf{x}^{(5)} = (5.0000000, 0.0000000, -0.5235988)^t$.

(b) With $\mathbf{x}^{(0)} = (1,1,1)^t$, we have $\mathbf{x}^{(9)} = (1.0364011, 1.0857072, 0.93119113)^t$.

(c) With $\mathbf{x}^{(0)} = (0,0,0.5)^t$, we have $\mathbf{x}^{(5)} = (0.00000000, 0.09999999, 1.0000000)^t$.

(d) With $\mathbf{x}^{(0)} = (0,0,0)^t$, we have $\mathbf{x}^{(5)} = (0.49814471, -0.19960600, -0.52882595)^t$.

8. (a) With

$$\mathbf{G}(\mathbf{x}) = \left(\sqrt{x_1 - x_2^2}, \sqrt{x_1^2 - x_2}\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (0.7, 0.4)^t,$$

we have $\mathbf{x}^{(14)} = (0.77184647, 0.41965131)^t$.

(b) With

$$\mathbf{G}(\mathbf{x}) = \left(x/\sqrt{3}, \sqrt{\left(1 + x_1^3\right)/(3x_1)}\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (0.4, 0.7)^t,$$

we have $\mathbf{x}^{(20)} = (0.4999980, 0.8660221)^t$.

(c) With

$$\mathbf{G}(\mathbf{x}) = \left(\sqrt{37 - x_2}, \sqrt{x_1 - 5}, 3 - x_1 - x_2\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (5, 1, -1)^t,$$

we have $\mathbf{x}^{(10)} = (6.0000002, 1.0000000, -3.9999971)^t$.

(d) With

$$\mathbf{G}(\mathbf{x}) = \left(\sqrt{2x_3 + x_2 - 2x_3^2}, \sqrt{\left(10x_3 + x_1^2\right)/8}, x_1^2/(7x_2)\right)^t \quad \text{and} \quad \mathbf{x}^{(0)} = (0.5, 0.5, 0)^t,$$

we have $\mathbf{x}^{(60)} = (0.5291548, 0.4000018, 0.09999853)^t$.

9. (a) With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, we have $\mathbf{x}^{(3)} = (0.5000000, 0, -0.5235988)^t$.

(b) With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, we have $\mathbf{x}^{(4)} = (1.036400, 1.085707, 0.9311914)^t$.

(c) With $\mathbf{x}^{(0)} = (0, 0, 0)^t$, we have $\mathbf{x}^{(3)} = (0, 0.1000000, 1.0000000)^t$.

(d) With $\mathbf{x}^{(0)} = (0, 0, 0)^t$, we have $\mathbf{x}^{(4)} = (0.4981447, -0.1996059, -0.5288260)^t$.

10. (a) Using $\mathbf{G}_1(\mathbf{x}) = (\sqrt{x_1 - {x_2}^2}, \sqrt{{x_1}^2} - x_2)^t$ and $\mathbf{x}^{(0)} = (0.7, 0.4)^t$ as in Exercise 8(a) gives a square root of a negative number as the first iteration. Thus, the method fails.

(b) Using $\mathbf{G}_1(\mathbf{x}) = \left(x/\sqrt{3}, \sqrt{(1 + {x_1}^3)/(3x_1)}\right)^t$ and $\mathbf{x}^{(0)} = (0.4, 0.7)^t$ as in Exercise 8(b) gives $\mathbf{x}^{(10)} = (0.49999807, 0.86602652)^t$. The convergence is accelerated for this problem.

(c) Using $\mathbf{G}_1(\mathbf{x}) = (\sqrt{37 - x_2}, \sqrt{x_1 - 5}, 3 - x_1 - x_2)^t$ and $\mathbf{x}^{(0)} = (5, 1, -1)^t$ as in Exercise 8(c) gives $\mathbf{x}^{(1)} = (6, 1, -4)^t$. The convergence very much accelerated for this problem.

(d) Using $\mathbf{G}_1(\mathbf{x}) = (\sqrt{2x_3 + x_2 - 2{x_2}^2}, \sqrt{(10x_3 + {x_1}^2)/8}, {x_1}^2/(2x_2))^t$ and $\mathbf{x}^{(0)} = (0.5, 0.5, 0)^t$ as in Exercise 8(d) leads to division by zero as the first iteration. Thus, the method fails.

11. A stable solution occurs when $x_1 = 8000$ and $x_2 = 4000$.

12. Let $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))^t$. Suppose $\mathbf{F}$ is continuous at $\mathbf{x_0}$. By Definition 10.3,

$$\lim_{\mathbf{x}\to\mathbf{x_0}} f_i(\mathbf{x}) = f_i(\mathbf{x}_0), \quad \text{for each } i = 1, ..., n.$$

Given $\epsilon > 0$, there exists $\delta_i > 0$ such that

$$|f_i(\mathbf{x}) - f_i(\mathbf{x}_0)| < \epsilon,$$

whenever $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta_i$ and $\mathbf{x} \in D$.

Let $\delta = \min_{1\le i\le n} \delta_i$. If $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$, then $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta_i$ and $|f_i(\mathbf{x}) - f_i(\mathbf{x}_0)| < \epsilon$, for each $i = 1, ..., n$, whenever $\mathbf{x} \in D$. This implies that

$$||\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)||_\infty < \epsilon,$$

whenever $||\mathbf{x} - \mathbf{x}_0|| < \delta$ and $\mathbf{x} \in D$. By the equivalence of vector norms, the result holds for all vector norms by suitably adjusting δ.

For the converse, let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$||\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)|| < \epsilon,$$

whenever $\mathbf{x} \in D$ and $||\mathbf{x} - \mathbf{x}_0|| < \delta$. By the equivalence of vector norms, a number $\delta' > 0$ can be found with

$$||f_i(\mathbf{x}) - f_i(\mathbf{x}_0)|| < \epsilon,$$

whenever $\mathbf{x} \in D$ and $||\mathbf{x} - \mathbf{x}_0|| < \delta'$.

Thus, $\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = f_i(\mathbf{x}_0)$, for $i = 1, ..., n$. Since $\mathbf{F}(\mathbf{x}_0)$ is defined, the conditions in Definition 10.3 hold, and $\mathbf{F}$ is continuous at $\mathbf{x}_0$.

13. When $A = O$, the zero matrix, the result follows immediately, because in this case $\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0) = \mathbf{0}$ for all $\mathbf{x}$ and $\mathbf{x}_0$ in $\mathbb{R}^n$.

Suppose $A \ne O$. Let $\mathbf{x}_0$ in $\mathbb{R}^n$ be arbitrary and $\varepsilon > 0$. Choose $\delta = \varepsilon/||A||$ and $\mathbf{x}_0 - \mathbf{x} < \delta$. Then

$$||F(\mathbf{x}) - F(\mathbf{x}_0)|| = ||A\mathbf{x} - A\mathbf{x}_0|| = ||A(\mathbf{x} - \mathbf{x}_0)||,$$

so

$$||F(\mathbf{x}) - F(\mathbf{x}_0)|| = ||A(\mathbf{x} - \mathbf{x}_0)|| \le ||A|| \cdot ||\mathbf{x} - \mathbf{x}_0|| < ||A|| \cdot \frac{\varepsilon}{||A||} = \varepsilon.$$

This, by Exercise 12, implies that $\mathbf{F}$ is continuous on $\mathbb{R}^n$.