1. (a) $s_1 = 1+\sqrt{2}$, $s_2 = -1+\sqrt{2}$
   (b) $s_1 = \sqrt{4+\sqrt{10}}$, $s_2 = \sqrt{4-\sqrt{10}}$
   (c) $s_1 = \sqrt{10}$, $s_2 = 2$
   (d) $s_1 = \sqrt{7}$, $s_2 = 1$, $s_3 = 1$

2. (a) $s_1 = \sqrt{2}$, $s_2 = \sqrt{2}$
   (b) $s_1 = 2$, $s_2 = 1$, $s_3 = 1$
   (c) $s_1 = \sqrt{5}$, $s_2 = \sqrt{3}$
   (d) $s_1 = \sqrt{5}$, $s_2 = \sqrt{2}$, $s_3 = 1$

3. (a)

$$U = \begin{bmatrix} -0.923880 & -0.382683 \\ -0.3826831 & 0.923880 \end{bmatrix}, \quad S = \begin{bmatrix} 2.414214 & 0 \\ 0 & 0.414214 \end{bmatrix},$$
$$V^t = \begin{bmatrix} -0.923880 & -0.382683 \\ -0.382683 & 0.923880 \end{bmatrix}$$

(b)

$$U = \begin{bmatrix} -0.824736 & 0.391336 & 0.408248 \\ -0.521609 & -0.247502 & -0.816497 \\ -0.218482 & -0.886340 & 0.408248 \end{bmatrix}. \quad S = \begin{bmatrix} 2.676243 & 0 \\ 0 & 0.915272 \\ 0 & 0 \end{bmatrix}.$$
$$V^t \approx \begin{bmatrix} -0.811242 & -0.584710 \\ -0.584710 & -0.8112420 \end{bmatrix}.$$

(c)

$$U = \begin{bmatrix} -0.632456 & -0.500000 & -0.522293 & -0.277867 \\ 0.316228 & -0.500000 & -0.301969 & 0.747539 \\ -0.316228 & -0.500000 & 0.797047 & 0.121309 \\ -0.632456 & 0.500000 & -0.027215 & 0.590982 \end{bmatrix}, \quad S = \begin{bmatrix} 3.162278 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$V^t = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(d)

$$U = \begin{bmatrix} -0.436436 & 0.707107 & 0.408248 & -0.377964 \\ 0.436436 & 0.707107 & -0.408248 & 0.377964 \\ -0.436436 & 0 & -0.816497 & -0.377964 \\ -0.654654 & 0 & 0 & 0.755929 \end{bmatrix}, \quad S = \begin{bmatrix} 2.645751 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$V^t = \begin{bmatrix} -0.577350 & -0.577350 & 0.577350 \\ 0 & 0.707107 & 0.707107 \\ 0.816497 & -0.408248 & 0.408248 \end{bmatrix}$$

4. (a)

$$U = \begin{bmatrix} -0.707107 & 0.707107 \\ 0.707107 & 0.707107 \end{bmatrix}, \quad S = \begin{bmatrix} 1.414214 & 0 \\ 0 & 1.414214 \end{bmatrix},$$

$$V^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)

$$U = \begin{bmatrix} -0.577350 & 0.408248 & 0.707107 \\ -0.577350 & 0.408248 & -0.707107 \\ -0.577350 & -0.816497 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$V^t = \begin{bmatrix} -0.577350 & -0.577350 & -0.577350 \\ 0.816497 & -0.408248 & -0.408248 \\ 0 & 0.707107 & -0.707107 \end{bmatrix}$$

(c)

$$U = \begin{bmatrix} -0.632456 & 0 & 0.258199 & -0.370901 & 0.629099 \\ 0 & -0.816497 & -0.430331 & -0.381832 & -0.048499 \\ 0.316228 & -0.408248 & 0.849731 & -0.075134 & -0.075134 \\ -0.316228 & -0.408248 & 0.010932 & 0.838799 & 0.172133 \\ 0.632456 & 0 & -0.161201 & 0.086066 & 0.752733 \end{bmatrix},$$

$$S = \begin{bmatrix} 2.236070 & 0 \\ 0 & 1.732051 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V^t = \begin{bmatrix} -0.707107 & 0.707107 \\ -0.707107 & -0.707107 \end{bmatrix}$$

(d)

$$U = \begin{bmatrix} -0.547723 & 0 & 0.707107 & -0.138916 & -0.425091 \\ -0.365148 & -0.408248 & 0 & -0.533212 & 0.644736 \\ -0.547723 & 0 & -0.707107 & -0.138916 & -0.425091 \\ -0.365148 & -0.408248 & 0 & 0.811044 & 0.205446 \\ -0.365148 & 0.816497 & 0 & -0.138916 & -0.425091 \end{bmatrix},$$

$$S = \begin{bmatrix} 2.236070 & 0 & 0 \\ 0 & 1.414214 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V^t = \begin{bmatrix} -0.408248 & -0.816497 & -0.408248 \\ 0.577350 & -0.577350 & 0.577350 \\ -0.707107 & 0 & 0.707107 \end{bmatrix}$$

5. For the matrix A in Example 2 we have

$$A^tA = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

So $A^tA(1,2,1)^t = (5,10,5)^t = 5(1,2,1)^t$, $A^tA(1,-1,1)^t = (2,-2,2)^t = 2(1,-1,1)^t$, and $A^tA(-1,0,1)^t = (-1,0,1)^t$.

6. The rank of a is the number of linearly independent rows in A, and the rank of A^t is the number of linearly independent row of A^t, which corresponds to the number of linearly independent columns of A. By Theorem 9.25 the number of linearly independent rows of a matrix is the same as the number of independent columns, so the rank of A is the same as the rank of A^t.

7. Let A be an $m \times n$ matrix. Theorem 9.25 implies that $\text{Rank}(A) = \text{Rank}(A^t)$, so $\text{Nullity}(A) = n - \text{Rank}(A)$ and $\text{Nullity}(A^t) = m - \text{Rank}(A^t) = m - \text{Rank}(A)$. Hence $\text{Nullity}(A) = \text{Nullity}(A^t)$ if and only if $n = m$.

8. The matrices S and S^t have nonzero values only on their diagonals, and the nonzero eigenvalues of A^tA and AA^t are the same, so the singular values of A^t are on the diagonal of S^t in decreasing order. In addition, the matrices U and V are both orthogonal, so a Singular Value Decomposition of $A^t$is given by

$$A^t = (U\,S\,V^t)^t = (V^t)^t\,S^t\,U^t = V\,S^t\,U^t$$

9. $\text{Rank}(S)$ is the number of nonzero entries on the diagonal of S. This corresponds to the number of nonzero eigenvalues (counting multiplicities) of A^tA. So $\text{Rank}(S) = \text{Rank}(A^tA)$, and by part (ii) of Theorem 9.26 this is the same as $\text{Rank}(A)$.

10. From Exercise 9 we know that $\text{Rank}(A) = \text{Rank}(S)$. Since A and S are both $m \times n$,

$$\text{Rank}(A) + \text{Nullity}(A) = n = \text{Rank}(S) + \text{Nullity}(S),$$

which implies that $\text{Nullity}(A) = \text{Nullity}(S)$.

11. The matrices U and V are orthogonal, so they are nonsingular with $U^{-1} = U^t$ and $V^{-1} = V^t$. Since

$$\det A = \det U \cdot \det S \cdot \det V^t,$$

with $\det U$ and $\det V^t$ both nonzero, $\det A = 0$ if and only if $\det S = 0$. Hence A in nonsingular if and only if S is nonsingular.

When A^{-1} exists we have the Singular Value Decomposition of A^{-1} given by

$$A^{-1} = \left(U\,S\,V^t\right)^{-1} = (V^t)^{-1}\,S^{-1}\,U^{-1} = \left(V^{-1}\right)^{-1}\,S^{-1}\,U^t = V\,S^{-1}\,U^t.$$

12. If A is $m \times n$, then A^t is $n \times m$, AA^t is $n \times n$, and A^tA is $m \times m$. So

$$n = \text{Rank}(A) + \text{Nullity}(A) = \text{Rank}(AA^t) + \text{Nullity}(AA^t) \quad \text{and} \quad m = \text{Rank}(A^tA) + \text{Nullity}(A^tA).$$

Since $\text{Rank}(A^tA) = \text{Rank}(A) = \text{Rank}(AA^t)$ we have

$$\text{Nullity}(A) = \text{Nullity}(A^tA) = \text{Nullity}(AA^t) \quad \text{if and only if} \quad m = n.$$

13. Yes. By Theorem 9.25 we have $\text{Rank}(A^tA) = \text{Rank}((A^tA)^t) = \text{Rank}(AA^t)$. Applying part (iii) of Theorem 9.26 gives $\text{Rank}(AA^t) = \text{Rank}(A^tA) = \text{Rank}(A)$.

14. Because P is orthogonal, we have $P^{-1} = P^t$, so $(PA)^t(PA) = A^t(P^tP)A = A^tA$. The singular values of A are the eigenvalues of A^tA, which must agree with those of $(PA)^t(PA)$. Hence the singular values of A and PA are the same.

15. The condition number is defined on page 470 as

$$K_2(A) = ||A||_2||A^{-1}||_2.$$

By Theorem 7.15 on page 447, we have $||A||_2^2 = \rho(A^tA)$, which is the largest eigenvalue of A^tA, that is, s_1^2.

In addition, Exercise 15 on page 450 states that the eigenvalues of the inverse of a nonsingular matrix are the reciprocals of the eigenvalues of the matrix, so

$$||A^{-1}||_2^2 = \rho((A^{-1})^tA^{-1}) = \rho((A^t)^{-1}A^{-1}) = \rho((AA^t)^{-1})$$

is the largest eigenvalue of $(AA^t)^{-1}$. This is the reciprocal of the smallest eigenvalue AA^t. By Theorem 9.26, the nonzero eigenvalues of A^tA and AA^t are the same, so this value is $1/s_n^2$. As a consequence,

$$K_2(A) = ||A||_2||A^{-1}||_2 = \frac{s_1}{s_n}.$$

16. For 1(a)the l_2 condition number is $\dfrac{1+\sqrt{2}}{-1+\sqrt{2}} = 3 + 2\sqrt{2}$, and for 1(d) it is $\sqrt{7}$. For 2(a) the l_2 condition number is 1, and for 2(b) it is 2.

17. (a) Use the tabulated values to construct

$$\mathbf{b} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 3.5 \\ 4.2 \\ 5.0 \\ 7.0 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1.0 \\ 1 & 2.0 \\ 1 & 3.0 \\ 1 & 4.0 \\ 1 & 5.0 \end{bmatrix}.$$

The matrix A has the singular value decomposition $A = U\,S\,V^t$, where

$$S = \begin{bmatrix} 7.691213 & 0 \\ 0 & 0.919370 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V^t = \begin{bmatrix} 0.266934 & 0.963715 \\ 0.963715 & -0.266934 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 0.160007 & 0.757890 & -0.414912 & -0.362646 & -0.310381 \\ 0.285308 & 0.467546 & 0.067225 & 0.399603 & 0.731982 \\ 0.410609 & 0.177202 & 0.837705 & -0.201287 & -0.240279 \\ 0.535909 & -0.113142 & -0.217438 & 0.654348 & -0.473867 \\ 0.661210 & -0.403486 & -0.272580 & -0.490018 & 0.292544 \end{bmatrix}$$

So

$$\mathbf{c} = U^t\mathbf{b} = \begin{bmatrix} 10.239160 \\ -0.024196 \\ 0.219013 \\ -0.076621 \\ 0.827743 \end{bmatrix},$$

and the components of $\mathbf{z}$ are

$$z_1 = \frac{c_1}{s_1} = \frac{10.239160}{7.691213} = 1.33, \quad \text{and} \quad z_2 = \frac{c_2}{s_2} = \frac{-0.024196}{0.919370} = -0.026,$$

This gives the least squares coefficients in $P_1(x) = a_0 + a_1 x$ as

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \mathbf{x} = V\,\mathbf{z} = \begin{bmatrix} 0.33 \\ 1.29 \end{bmatrix},$$

that is, $P_1(x) = 0.33 + 1.29x$.

(b) We have the same vector **b** as in part(a) but the matrix a is now

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}.$$

A Singular Value Decomposition of A is

$$S = \begin{bmatrix} 32.15633 & 0 & 0 \\ 0 & 2.197733 & 0 \\ 0 & 0 & 0.374376 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad V^t = \begin{bmatrix} -0.055273 & -0.224442 & -0.972919 \\ -0.602286 & -0.769677 & 0.211773 \\ 0.796364 & -0.597681 & 0.092637 \end{bmatrix}$$

and

$$U = \begin{bmatrix} -0.038954 & -0.527903 & 0.778148 & -0.008907 & -0.337944 \\ -0.136702 & -0.589038 & -0.075997 & 0.243571 & 0.754483 \\ -0.294961 & -0.457453 & -0.435258 & -0.677268 & -0.235783 \\ -0.513732 & -0.133148 & -0.299632 & 0.659453 & -0.440105 \\ -0.793015 & 0.383877 & 0.330878 & -0.216849 & 0.259350 \end{bmatrix}$$

We now find the vector $\mathbf{c} = U^t\mathbf{b}$, construct $\mathbf{z}$ by dividing the components of the vector $\mathbf{c}$ by the three singular values. Then the coefficients of the least squares polynomial are given by the components of the vector $V\mathbf{z}$. This produces

$$P_2(x) = 0.18 + 1.418571x - 0.0214286x^2.$$

18. (a) Use the tabulated values to construct

$$\mathbf{b} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix} = \begin{bmatrix} 1 & 1.0 & 1.0 \\ 1 & 1.1 & 1.21 \\ 1 & 1.3 & 1.69 \\ 1 & 1.5 & 2.25 \\ 1 & 1.9 & 3.61 \\ 1 & 2.1 & 4.41 \end{bmatrix}.$$

The matrix A has the singular value decomposition $A = U\,S\,V^t$, where

$$U = \begin{bmatrix} -0.203339 & -0.550828 & 0.554024 & 0.055615 & -0.177253 & -0.560167 \\ -0.231651 & -0.498430 & 0.185618 & 0.165198 & 0.510822 & 0.612553 \\ -0.294632 & -0.369258 & -0.337742 & -0.711511 & -0.353683 & 0.177288 \\ -0.366088 & -0.20758 & -0.576499 & 0.642950 & -0.264204 & -0.085730 \\ -0.534426 & 0.213281 & -0.200202 & -0.214678 & 0.628127 & -0.433808 \\ -0.631309 & 0.472467 & 0.414851 & 0.062426 & -0.343809 & 0.289864 \end{bmatrix},$$

$$S = \begin{bmatrix} 7.844127 & 0 & 0 \\ 0 & 1.223790 & 0 \\ 0 & 0 & 0.070094 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad V^t = \begin{bmatrix} -0.288298 & -0.475702 & -0.831018 \\ -0.768392 & -0.402924 & 0.497218 \\ 0.571365 & -0.781895 & 0.249363 \end{bmatrix}.$$

So

$$\mathbf{c} = U^t\mathbf{b} = \begin{bmatrix} -5.955009 \\ -1.185591 \\ -0.044985 \\ -0.003732 \\ -0.000493 \\ -0.001963 \end{bmatrix},$$

and the components of $\mathbf{z}$ are

$$z_1 = \frac{c_1}{s_1} = \frac{-5.955009}{7.844127} = -0.759168, \quad z_2 = \frac{c_2}{s_2} = \frac{-1.185591}{1.223790} = -0.968786,$$

and

$$z_3 = \frac{c_3}{s_3} = \frac{-0.044985}{0.070094} = -0.641784.$$

This gives the least squares coefficients in $P_2(x) = a_0 + a_1x + a_2x^2$ as

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \mathbf{x} = V\,\mathbf{z} = \begin{bmatrix} 0.596581 \\ 1.253293 \\ -0.010853 \end{bmatrix}.$$

The least squares error using these values uses the last three components of $\mathbf{c}$, and is

$$||A\mathbf{x} - \mathbf{b}||_2 = \sqrt{c_4^2 + c_5^2 + c_6^2} = \sqrt{(-0.003732)^2 + (-0.000493)^2 + (-0.001963)^2} = 0.004244.$$

(b) Use the tabulated values to construct

$$\mathbf{b} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \\ 1 & x_5 & x_5^2 & x_5^3 \end{bmatrix} = \begin{bmatrix} 1 & 1.0 & 1.0 & 1.0 \\ 1 & 1.1 & 1.21 & 1.331 \\ 1 & 1.3 & 1.69 & 2.197 \\ 1 & 1.5 & 2.25 & 3.375 \\ 1 & 1.9 & 3.61 & 6.859 \\ 1 & 2.1 & 4.41 & 9.261 \end{bmatrix}.$$

The matrix A has the singular value decomposition $A = U\,S\,V^t$, where

$$U = \begin{bmatrix} -0.116086 & -0.514623 & 0.569113 & -0.437866 & -0.381082 & 0.246672 \\ -0.143614 & -0.503586 & 0.266325 & 0.184510 & 0.535306 & 0.578144 \\ -0.212441 & -0.448121 & -0.238475 & 0.484990 & 0.180600 & -0.655247 \\ -0.301963 & -0.339923 & -0.549619 & 0.038581 & -0.573591 & 0.400867 \\ -0.554303 & 0.074101 & -0.306350 & -0.636776 & 0.417792 & -0.115640 \\ -0.722727 & 0.399642 & 0.390359 & 0.363368 & -0.179026 & 0.038548 \end{bmatrix},$$

$$S = \begin{bmatrix} 14.506808 & 0 & 0 & 0 \\ 0 & 2.084909 & 0 & 0 \\ 0 & 0 & 0.198760 & 0 \\ 0 & 0 & 0 & 0.868328 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$V^t = \begin{bmatrix} -0.141391 & -0.246373 & -0.449207 & -0.847067 \\ -0.639122 & -0.566437 & -0.295547 & 0.428163 \\ 0.660862 & -0.174510 & -0.667840 & 0.294610 \\ -0.367142 & 0.766807 & -0.514640 & 0.111173 \end{bmatrix}.$$

So

$$\mathbf{c} = U^t \mathbf{b} = \begin{bmatrix} -5.632309 \\ -2.268376 \\ 0.036241 \\ 0.005717 \\ -0.000845 \\ -0.004086 \end{bmatrix},$$

and the components of $\mathbf{z}$ are

$$z_1 = \frac{c_1}{s_1} = \frac{-5.632309}{14.506808} = -0.388253, \quad z_2 = \frac{c_2}{s_2} = \frac{-2.268376}{2.084909} = -1.087998,$$
$$z_3 = \frac{c_3}{s_3} = \frac{0.036241}{0.198760} = 0.182336, \quad \text{and} \quad z_4 = \frac{c_4}{s_4} = \frac{0.005717}{0.868328} = 0.65843.$$

This gives the least squares coefficients in $P_2(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ as

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{x} = V\,\mathbf{z} = \begin{bmatrix} 0.629019 \\ 1.185010 \\ 0.035333 \\ -0.010047 \end{bmatrix}.$$

The least squares error using these values uses the last two components of $\mathbf{c}$, and is

$$||A\mathbf{x} - \mathbf{b}||_2 = \sqrt{c_5^2 + c_6^2} = \sqrt{(-0.000845)^2 + (-0.004086)^2} = 0.004172.$$