1. Two iterations of the QR Algorithm without shifting produce the following matrices.

(a) $$A^{(3)} = \begin{bmatrix} 3.142857 & -0.559397 & 0.0 \\ -0.559397 & 2.248447 & -0.187848 \\ 0.0 & -0.187848 & 0.608696 \end{bmatrix}$$

(b) $$A^{(3)} = \begin{bmatrix} 4.549020 & 1.206958 & 0.0 \\ 1.206958 & 3.519688 & 0.000725 \\ 0.0 & 0.000725 & -0.068708 \end{bmatrix}$$

(c) $$A^{(3)} = \begin{bmatrix} 4.592920 & -0.472934 & 0.0 \\ -0.472934 & 3.108760 & -0.232083 \\ 0.0 & -0.232083 & 1.298319 \end{bmatrix}$$

(d) $$A^{(3)} = \begin{bmatrix} 3.071429 & 0.855352 & 0.0 & 0.0 \\ 0.855352 & 3.314192 & -1.161046 & 0.0 \\ 0.0 & -1.1610446 & 3.331770 & 0.268898 \\ 0.0 & 0.0 & 0.268898 & 0.282609 \end{bmatrix}$$

(e) $$A^{(3)} = \begin{bmatrix} -3.607843 & 0.612882 & 0.0 & 0.0 \\ 0.612882 & -1.395227 & -1.111027 & 0.0 \\ 0.0 & -1.111027 & 3.133919 & 0.346353 \\ 0.0 & 0.0 & 0.346353 & 0.869151 \end{bmatrix}$$

(f) $$A^{(3)} = \begin{bmatrix} 1.013260 & 0.279065 & 0.0 & 0.0 \\ 0.279065 & 0.696255 & 0.107448 & 0.0 \\ 0.0 & 0.107448 & 0.843061 & 0.310832 \\ 0.0 & 0.0 & 0.310832 & 0.317424 \end{bmatrix}$$

2. Two iterations of the QR Algorithm without shifting produce the following matrices.

(a) $$A^{(3)} = \begin{bmatrix} 2.63333333 & -1.16856988 & 0 \\ -1.16856988 & 0.93786276 & -2.57594498 \\ 0 & -2.57594498 & 0.42880391 \end{bmatrix}$$

(b) $$A^{(3)} = \begin{bmatrix} 4.60130719 & 1.38545134 & 0 \\ 1.38545134 & 4.16532313 & 0.24280011 \\ 0 & 0.24280011 & 1.23336968 \end{bmatrix}$$

(c) $$A^{(3)} = \begin{bmatrix} 6.28571429 & 1.16057692 & 0 & 0 & 0 \\ 1.16057692 & 5.26984127 & 1.49897084 & 0 & 0 \\ 0 & 1.49897084 & 4.80808081 & 1.50776756 & 0 \\ 0 & 0 & 1.50776756 & 3.07260525 & 0.23213209 \\ 0 & 0 & 0 & 0.23213209 & 0.56375839 \end{bmatrix}$$

(d) $$A^{(3)} = \begin{bmatrix} 5.58655992 & -0.60565234 & 0 & 0 & 0 \\ -0.60565234 & 3.92585374 & 0.02385124 & 0 & 0 \\ 0 & 0.02385124 & 3.03035281 & -1.22483017 & 0 \\ 0 & 0 & -1.22483017 & 2.29033145 & 0.73675527 \\ 0 & 0 & 0 & 0.73675527 & 1.66690207 \end{bmatrix}$$

3. The matrices in Exercise 1 have the following eigenvalues, accurate to within 10^{-5}.

   (a) 3.414214, 2.000000, 0.58578644
   (b) $-0.06870782, 5.346462, 2.722246$
   (c) 1.267949, 4.732051, 3.000000
   (d) 4.745281, 3.177283, 1.822717, 0.2547188
   (e) $3.438803, 0.8275517, -1.488068, -3.778287$
   (f) 0.9948440, 1.189091, 0.5238224, 0.1922421

4. The matrices have the following eigenvalues, accurate to within 10^{-5}.

   (a) $3.9115033, 2.1294613, -2.0409646$
   (b) $1.2087122, 5.7912878, 3.0000000$
   (c) $6.0000000, 2.0000000, 4.0000000, 7.4641016, 0.5358984$
   (d) $4.0274350, 2.0707128, 3.7275564, 5.7839956, 0.8903002$

5. The matrices in Exercise 1 have the following eigenvectors, accurate to within 10^{-5}.

   (a) $(-0.7071067, 1, -0.7071067)^t$, $(1, 0, -1)^t$, $(0.7071068, 1, 0.7071068)^t$
   (b) $(0.1741299, -0.5343539, 1)^t$, $(0.4261735, 1, 0.4601443)^t$, $(1, -0.2777544, -0.3225491)^t$
   (c) $(0.2679492, 0.7320508, 1)^t$, $(1, -0.7320508, 0.2679492)^t$, $(1, 1, -1)^t$
   (d) $(-0.08029447, -0.3007254, 0.7452812, 1)^t$, $(0.4592880, 1, -0.7179949, 0.8727118)^t$, $(0.8727118, 0.7179949, 1, -0.4592880)^t$ $(1, -0.7452812, -0.3007254, 0.08029447)^t$
   (e) $(-0.01289861, -0.07015299, 0.4388026, 1)^t$, $(-0.1018060, -0.2878618, 1, -0.4603102)^t$, $(1, 0.5119322, 0.2259932, -0.05035423)^t$ $(-0.5623391, 1, 0.2159474, -0.03185871)^t$
   (f) $(-0.1520150, -0.3008950, -0.05155956, 1)^t$, $(0.3627966, 1, 0.7459807, 0.3945081)^t$, $(1, 0.09528962, -0.6907921, 0.1450703)^t$, $(0.8029403, -0.9884448, 1, -0.1237995)^t$

6. (a) The inverse power method using $\mathbf{x}^{(0)} = (1,1,1)^t$ gives the following eigenvalues and eigenvectors.

$\lambda_1 = 3.91150331, \quad \mathbf{x}^{(9)} = (0.34132546, -0.51819891, 1)^t$

$\lambda_2 = 2.12946128, \quad \mathbf{x}^{(5)} = (1, -0.17819414, -0.21683219)^t$

$\lambda_3 = -2.04096459, \quad \mathbf{x}^{(6)} = (0.27053411, 1, 0.21292940)^t$

(b) The inverse power method using $\mathbf{x}^{(0)} = (1,1,1)^t$ gives the following eigenvalues and eigenvectors.

$\lambda_1 = 1.20871215, \quad \mathbf{x}^{(2)} = (0.5, -0.89564392, 1)^t$

$\lambda_2 = 5.79128785, \quad \mathbf{x}^{(2)} = (0.35825757, 1, 0.71654514)^t$

$\lambda_3 = 2.99999999, \quad \mathbf{x}^{(2)} = (1, 0, -0.5)^t$

(c) The inverse power method using $\mathbf{x}^{(0)} = (1,1,1,1)^t$ gives the following eigenvalues and eigenvectors.

$\lambda_1 = 5.99999999, \quad \mathbf{x}^{(5)} = (1, 1, 0, -1, -1)^t$

$\lambda_2 = 1.99999999, \quad \mathbf{x}^{(5)} = (1, -1, 0, 1, -1)^t$

$\lambda_3 = 3.99999999, \quad \mathbf{x}^{(2)} = (1, 0, -1, 0, 1)^t$

$\lambda_4 = 7.46410162, \quad \mathbf{x}^{(2)} = (0.5, 0.86602540, 1, 0.86602540, 0.5)^t$

$\lambda_5 = 0.53589838, \quad \mathbf{x}^{(2)} = (0.5, -0.86602540, 1, -0.86602540, 0.5)^t$

(d) The inverse power method using $\mathbf{x}^{(0)} = (1,1,1,1)^t$ gives the following eigenvalues and eigenvectors.

$\lambda_1 = 4.02743496, \quad \mathbf{x}^{(2)} = (-0.5009008, -0.4871586, -0.13534334, 1, 0.97329762)^t$

$\lambda_2 = 2.0707128, \quad \mathbf{x}^{(2)} = (-0.01115300, -0.03267035, 0.34106327, -0.92928720, 1)^t$

$\lambda_3 = 3.72755642, \quad \mathbf{x}^{(2)} = (0.78588946, 1, 0.06722944, 0.04156975, 0.05713611)^t$

$\lambda_4 = 5.78399557, \quad \mathbf{x}^{(2)} = (1, -0.78399557, -0.03323416, 0.00548238, 0.00196925)^t$

$\lambda_5 = 0.89030025, \quad \mathbf{x}^{(2)} = (-0.01445632, -0.05941112, 1, 0.24454382, -0.11591404)^t$

7. (a) First note that for any vector $\mathbf{x} = (x_1, x_2)^t$ we have

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix},$$

and that

$$\begin{aligned}(x_1\cos\theta - x_2\sin\theta)^2 + (x_1\sin\theta + x_2\cos\theta)^2 &= x_1^2((\cos\theta)^2 + (\sin\theta)^2) + x_2^2((-\sin\theta)^2 + (\cos\theta)^2) \\ &= x_1^2 + x_2^2.\end{aligned}$$

So the l_2 norms are the same.
Now let $\mathbf{z} = (z_1, z_2)^t$ represent the vector that has the same magnitude as $\mathbf{x}$ but has been rotated by the angle θ. Let ϕ be the angle from the x-axis to the point (x_1, x_2) and let $r = \sqrt{x_1^2 + x_2^2} = ||\mathbf{x}||_2$. Then

$$x_1 = r\cos\phi \quad \text{and} \quad x_2 = r\sin\phi.$$

In a similar manner,

$$\begin{aligned} z_1 &= r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta = x_1\cos\theta - x_2\sin\theta \\ z_2 &= r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta = x_1\sin\theta + x_2\cos\theta \end{aligned}$$

So the unique vector $\mathbf{z}$ that has the same l_2 norm as $\mathbf{x}$ and is rotated by an angle of θ is given by multiplying $\mathbf{x}$ by the rotation matrix.

(b) Consider the the vector $\mathbf{x} = (1, 1)^t$, which has l_∞ norm 1. If $\theta = \pi/4$, then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix},$$

which has l_∞ norm $\sqrt{2}$.

8. Let $P = (p_{ij})$ be a rotation matrix with nonzero entries $p_{jj} = p_{ii} = \cos\theta$, $p_{ij} = -p_{ji} = \sin\theta$, and $p_{kk} = 1$, if $k \neq i$ and $k \neq j$. For any $n \times n$ matrix A,

$$(AP)_{rs} = \sum_{k=1}^{n} a_{rk} p_{ks}.$$

If $s \neq i, j$, then $p_{ks} = 0$ unless $k = s$. Thus, $(AP)_{rs} = a_{rs}$.
If $s = j$, then

$$(AP)_{rj} = a_{rj}p_{jj} + a_{ri}p_{ij} = a_{rj}\cos\theta + a_{ri}\sin\theta.$$

If $s = i$, then

$$(AP)_{ri} = a_{rj}p_{ji} + a_{ri}p_{ii} = -a_{rj}\sin\theta + a_{ri}\cos\theta.$$

Similarly, $(PA)_{rs} = \sum_{k=1}^{n} p_{rk}a_{ks}$. If $r \neq i, j$, then $p_{rk} = 0$ unless $r = k$. Thus, $(PA)_{rs} = a_{rs}$.
If $r = i$, then

$$(PA)_{is} = p_{ij}a_{js} + p_{ii}a_{is} = a_{js}\sin\theta + a_{is}\cos\theta.$$

If $r = j$, then

$$(PA)_{js} = p_{jj}a_{js} + p_{ji}a_{is} = a_{js}\cos\theta - a_{is}\sin\theta.$$

9. Let $C = RQ$, where R is upper triangular and Q is upper Hessenberg. Then $c_{ij} = \sum_{k=1}^{n} r_{ik}q_{kj}$. Since R is an upper triangular matrix, $r_{ik} = 0$ if $k < i$. Thus $c_{ij} = \sum_{k=i}^{n} r_{ik}q_{kj}$. Since Q is an upper Hessenberg matrix, $q_{kj} = 0$ if $k > j + 1$. Thus, $c_{ij} = \sum_{k=i}^{j+1} r_{ik}q_{kj}$. The sum will be zero if $i > j + 1$. Hence, $c_{ij} = 0$ if $i \geq j + 2$. This means that C is an upper Hessenberg matrix.

10. (a) We have

$$P_2^t P_3^t = \begin{bmatrix} c_2 & -s_2 & 0 & 0 & \dots & 0 \\ s_2 & c_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & c_3 & -s_3 & 0 & \dots & 0 \\ 0 & s_3 & c_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_2 & -s_2c_3 & s_2s_3 & 0 & \dots & 0 \\ s_2 & c_2c_3 & -s_3c_2 & 0 & \dots & 0 \\ 0 & s_3 & c_3 & 0 & & \vdots \\ 0 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

(b) Let $Q_k = P_2^t P_3^t \cdots P_k^t$ be an upper triangular matrix except for the entries $(Q_k)_{2,1}$, $(Q_k)_{3,2}$, $\dots$, $(Q_k)_{k,k-1}$, which may be nonzero. Since multiplying Q_k by the rotation matrix $(P_{k+1})^t$ can only change columns k and $k+1$ in forming $Q_{k+1} = P_2^t P_3^t \cdots P_k^t P_{k+1}^t$, we only need to consider the entries $(Q_{k+1})_{i,k}$ and $(Q_{k+1})_{i,k+1}$, for $i = k+2, \dots, n$. First, we have

$$(Q_{k+1})_{i,k} = \sum_{j=1}^{n} (Q_k)_{i,j} \left(P_{k+1}^t\right)_{j,k} = (Q_k)_{i,k} c_{k+1} + (Q_k)_{i,k-1} s_{k+1}.$$

However, $(Q_k)_{i,k} = 0$ for $i > k$ and $(Q_k)_{i,k+1} = 0$, for $i > k+1$. Thus, $(Q_{k+1})_{i,k} = 0$, for $i \geq k+2$. Further,

$$(Q_{k+1})_{i,k+1} = -(Q_k)_{i,k} s_{k+1} + (Q_k)_{i,k+1} c_{k+1} = 0,$$

for $i \geq k+2$. Thus, $Q_{k+1} = P_2^t P_3^t \cdots P_{k+1}^t$ is upper triangular except for the entries in positions $(2,1)$, $(3,2)$, $\dots$, $(k, k-1)$, $(k+1, k)$, which may be nonzero.

(c) From parts (a) and (b) and mathematical induction, it follows that the matrix $P_2^t P_3^t \cdots P_n^t$ is upper triangular except that the entries in positions $(2,1)$, $(3,2)$, $\dots$, $(n, n-1)$ may be nonzero. Thus, the entries in positions (i,j), for $i \geq j+2$ are zero, which means that $P_2^t P_3^t \cdots P_n^t$ is an upper Hessenberg matrix.

11. The following algorithm implements Jacobi's method for symmetric matrices.

INPUT: dimension n, matrix $A = (a_{ij})$, tolerance TOL, maximum number of iterations N.
OUTPUT: eigenvalues $\lambda_1, \ldots, \lambda_n$ of A or a message that the number of iterations was exceeded.

STEP 1 Set $FLAG = 1$; $k1 = 1$.

STEP 2 While $(FLAG = 1)$ do Steps 3 – 10

STEP 3 For $i = 2, ..., n$ do Steps 4 – 8.

STEP 4 For $j = 1, ..., i-1$ do Steps 5 – 8.

STEP 5 If $a_{ii} = a_{jj}$ then set
$\mathrm{CO} = 0.5\sqrt{2}$;
$\mathrm{SI} = \mathrm{CO}$
else set
$b = |a_{ii} - a_{jj}|$;
$c = 2a_{ij}\ \mathrm{sign}(a_{ii} - a_{jj})$;
$\mathrm{CO} = 0.5\left(1 + b/\left(c^2 + b^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$;
$\mathrm{SI} = 0.5c/\left(CO\left(c^2 + b^2\right)^{\frac{1}{2}}\right)$.

STEP 6 For $k = 1, ..., n$
if $(k \neq i)$ and $(k \neq j)$ then
set $x = a_{k,j}$;
$y = a_{k,i}$;
$a_{k,j} = \mathrm{CO} \cdot x + \mathrm{SI} \cdot y$;
$a_{k,i} = \mathrm{CO} \cdot y + \mathrm{SI} \cdot x$;
$x = a_{j,k}$;
$y = a_{i,k}$;
$a_{j,k} = \mathrm{CO} \cdot x + \mathrm{SI} \cdot y$;
$a_{i,k} = \mathrm{CO} \cdot y - \mathrm{SI} \cdot x$.

STEP 7 Set $x = a_{j,j}$;
$y = a_{i,i}$;
$a_{j,j} = \mathrm{CO} \cdot \mathrm{CO} \cdot x + 2 \cdot \mathrm{SI} \cdot CO \cdot a_{j,i} + \mathrm{SI} \cdot \mathrm{SI} \cdot y$;
$a_{i,i} = \mathrm{SI} \cdot \mathrm{SI} \cdot x - 2 \cdot \mathrm{SI} \cdot \mathrm{CO} \cdot a_{i,j} + \mathrm{CO} \cdot \mathrm{CO} \cdot y$.

STEP 8 Set $a_{i,j} = 0$; $a_{j,i} = 0$.

STEP 9 Set

$$s = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

STEP 10 If $s < TOL$ then for $i = 1, ..., n$ set
$\lambda_i = a_{ii}$;
OUTPUT $(\lambda_1, ..., \lambda_n)$;
set $FLAG = 0$.
else set $k1 = k1 + 1$;
if $k1 > N$ then set $FLAG = 0$.

STEP 11 If $k1 > N$ then
OUTPUT ('Maximum number of iterations exceeded');
STOP.

12. Jacobi's method produces the following eigenvalues, accurate to within the tolerance:

(a) 3.414214, 0.5857864, 2.0000000; 3 iterations

(b) 2.722246, 5.346462, −0.06870782; 3 iterations

(c) 4.732051, 3, 1.267949; 3 iterations

(d) 0.2547188, 1.822717, 3.177283, 4.745281; 3 iterations

(e) −1.488068, −3.778287, 0.8275517, 3.438803; 3 iterations

(f) 0.1922421, 1.189091, 0.5238224, 0.9948440; 3 iterations

13. (a) To within 10^{-5}, the eigenvalues are $2.618034, 3.618034, 1.381966$, and 0.3819660.

(b) In terms of p and ρ the eigenvalues are $-65.45085p/\rho$, $-90.45085p/\rho$, $-34.54915p/\rho$, and $-9.549150p/\rho$.

14. (a) When $\alpha = 1/4$, we have 0.97972997, 0.92060076, 0.82741863, 0.70771852, 0.57114328, 0.42886719, 0.29232093, 0.17255567, 0.07939063, and 0.02025441.

(b) When $\alpha = 1/2$, we have 0.95945994, 0.84120152, 0.65483725, 0.41543703, 0.14228657, −0.14226561, −0.41535813, −0.65488866, −0.84121873, and −0.95949118.

(c) When $\alpha = 3/4$, we have 0.93918991, 0.76180227, 0.48225588, 0.12315555, −0.28657015, −0.71339842, −1.12303720, −1.48233299, −1.76182810, and −1.93923676.
The method appears to be stable for $\alpha \leq \frac{1}{2}$.

15. The actual eigenvalues are as follows:

(a) When $\alpha = \frac{1}{4}$ we have 0.97974649, 0.92062677, 0.82743037, 0.70770751, 0.57115742, 0.42884258, 0.29229249, 0.17256963, 0.07937323, and 0.02025351.

(b) When $\alpha = \frac{1}{2}$ we have 0.95949297, 0.84125353, 0.65486073, 0.41541501, 0.14231484, −0.14231484, −0.41541501, −0.65486073, −0.84125353, and −0.95949297.

(c) When $\alpha = \frac{3}{4}$ we have 0.93923946, 0.76188030, 0.48229110, 0.12312252, −0.28652774, −0.71347226, −1.12312252, −1.48229110, −1.76188030, and −1.93923946. The method appears to be stable for $\alpha \leq \frac{1}{2}$.