1. In each instance we will compare the characteristic polynomial of A $(C(A))$ to that of B. They must agree if the matrices are to be similar.

   (a) $p(A) = x^2 - 4x + 3 \neq x^2 - 2x - 3 = p(B)$.
   (b) $p(A) = x^2 - 5x + 6 \neq x^2 - 6x + 6 = p(B)$.
   (c) $p(A) = x^3 - 4x^2 + 5x - 2 \neq x^3 - 4x^2 + 5x - 6 = p(B)$.
   (d) $p(A) = x^3 - 5x^2 + 12x - 11 \neq x^3 - 4x^2 + 4x + 11 = p(B)$.

2. For a pair of matrices to be similar the determinants and characteristic polynomials must be the same.

   (a) $\det(A) = 3 \neq 2 = \det(B)$
   (b) $\det(A) = -4 = \det(B)$, but $p(A) = \lambda^2 + \lambda - 4 \neq \lambda^2 - \lambda - 4 = p(B)$.
   (c) $\det(A) = 1 \neq -8 = \det(B)$
   (d) $\det(A) = -24 \neq 0 = \det(B)$

3. In each case we have $A^3 = (PDP^{(-1)})(PDP^{(-1)})(PDP^{(-1)}) = PD^3P^{(-1)}$.

   (a) $\begin{bmatrix} \frac{26}{5} & -\frac{14}{5} \\ -\frac{21}{5} & \frac{19}{5} \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 9 \\ 0 & -8 \end{bmatrix}$
   (c) $\begin{bmatrix} \frac{9}{5} & -\frac{8}{5} & \frac{7}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} & -\frac{6}{5} \end{bmatrix}$
   (d) $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

4. (a) The technique is described in part (b). The result is

$$\begin{bmatrix} 10 & -6 \\ -9 & 7 \end{bmatrix}.$$

(b) It would be easy to simply use A^3 from Exercise 3(b) and multiply by A. However, we don't explicitly have A. Alternatively,

$$A^4 = PD^4P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -15 \\ 0 & 16 \end{bmatrix}.$$

(c) The technique is described in part (b). The result is

$$A^4 = \frac{1}{5}\begin{bmatrix} 7 & -4 & 1 \\ 4 & -3 & 2 \\ 2 & -4 & 6 \end{bmatrix}.$$

(d) The technique is described in part (b). The result is

$$A^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}.$$

5. They are all diagonalizable with P and D as follows.

(a) $P = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $P = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1+\sqrt{2} & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. (a) $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $P = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

7. Only the matrices in parts (a) and (c) are positive definite.

(a) $Q = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8. The matrix will be positive definite if and only if the all the principle leading submatrices have a positive determinant. Let A_n denote the $n \times n$ principle leading submatrix of A.

(a) $\det(A_1) = 4 \quad \det(A_2) = 12,$ and $\det(A) = 44$, so A is positive definite. A factorization $D = Q^t AQ$ has

$$Q = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{5}}{4} & \frac{\sqrt{10}}{5} & \frac{\sqrt{10}}{5} \\ -\frac{\sqrt{5}}{2} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4+\sqrt{5} & 0 \\ 0 & 0 & 4-\sqrt{5} \end{bmatrix}$$

(b) $\det(A) = 0$ so A is not positive definite.

(c) $\det(A) = -5$ so A is not positive definite.

(d) $\det(A_1) = 8 \quad \det(A_2) = 48, \quad \det(A_3) = 352,$ and $\det(A) = 2736$, so A is positive definite. A factorization $D = Q^t AQ$ has, approximately,

$$Q^t = \begin{bmatrix} -0.6005 & -0.6005 & -0.4496 & -0.2770 \\ -0.7071 & 0.7071 & 0 & 0 \\ 0.2844 & 0.2844 & -0.8906 & 0.2122 \\ -0.2419 & -0.2419 & 0.0688 & 0.9371 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 13.9587 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6.4841 & 0 \\ 0 & 0 & 0 & 7.5572 \end{bmatrix}$$

9. In each case the matrix fails to have 3 linearly independent eigenvectors.

   (a) $\det(A) = 12$, so A is nonsingular.
   (b) $\det(A) = -1$, so A is nonsingular.
   (c) $\det(A) = 12$, so A is nonsingular.
   (d) $\det(A) = 1$, so A is nonsingular.

10. (a) The matrix is clearly singular because it has a row (and a column) all of whose entries are 0. However, the eigenvalues of A, which are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 3$ are distinct so the corresponding eigenvectors, $\mathbf{x}_1 = (0, 0, 1)^t$, $\mathbf{x}_1 = (1, 1, 0)^t$, and $\mathbf{x}_3 = (1, -1,)^t$, can be used to form the columns of the matrix P with $A = PDP^{-1}$. The matrix D is the diagonal matrix with diagonal entries in the order 0, 1, and 3.

    (b) The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 3$, so A is singular. However, there are three linearly independent eigenvectors of A. The eigenvector $\mathbf{x}_1 = (1, 1, 1)^t$, corresponding to $\lambda_1 = 0$, and the eigenvectors $\mathbf{x}_2 = (1, 0, -1)^t$ and $\mathbf{x}_3 = (1, -1, 0)^t$, corresponding to $\lambda_2 = \lambda_3 = 3$. These eigenvectors can be used to form the columns of the matrix P with $A = PDP^{-1}$. The matrix D is the diagonal matrix with diagonal entries in the order 0, 3, and 3.

11. (a) The eigenvalues and associated eigenvectors are
    $\lambda_1 = 5.307857563$, $(0.59020967, 0.51643129, 0.62044441)^t$;
    $\lambda_2 = -0.4213112993$, $(0.77264234, -0.13876278, -0.61949069)^t$;
    $\lambda_3 = -0.1365462647$, $(0.23382978, -0.84501102, 0.48091581)^t$.

    (b) A is not positive definite because $\lambda_2 < 0$ and $\lambda_3 < 0$.

12. Since B is nonsingular, B^{-1} exists and

$$AB = I \cdot (AB) = (B^{-1}B)(AB) = B^{-1}(BA)B.$$

    So AB is similar to BA.

13. Because A is similar to B and B is similar to C there exist invertible matrices S and T with $A = S^{-1}BS$ and $B = T^{-1}CT$ Hence A is similar to C because

$$A = S^{-1}BS = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS).$$

14. Suppose that $A = PBP^{-1}$.

(a) We have

$$\begin{aligned}\det(A) =& \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1})\\ =& \det(P)\det(P^{-1})\det(B) = \det(PP^{-1})\det(B) = \det(I)\det(B) = \det(B).\end{aligned}$$

(b) We have

$$\begin{aligned}p(A) =& \det(A - \lambda I) = \det(PBP^{-1} - \lambda P \cdot I \cdot P^{-1})\\ =& \det(P)\det(B - \lambda I)\det(P^{-1}) = \det(B - \lambda I) = p(B).\end{aligned}$$

(c) The characteristic polynomials of A and B agree, so A and B have the same eigenvalues. The matrix A is singular if and only if 0 is an eigenvalue of A, which is true if and only if 0 is an eigenvalue of B. So A is nonsingular if and ony if B is nonsingular.

(d) We have

$$A^{-1} = (PBP^{-1})^{-1} = (P^{-1})^{-1}B^{-1}P^{-1} = PB^{-1}P^{-1}.$$

So A^{-1} is similar to B^{-1}.

(e) We have

$$A^t = (PBP^{-1})^t = (P^{-1})^t B^t P^t = (P^t)^{-1}B^t P^t.$$

Since P is invertible if and only if P^t is invertible, A^t is similar to B^t.

15. The matrix A has an eigenvalue of multiplicity 1 at $\lambda_1 = 3$ with eigenvector $\mathbf{s}_1 = (0, 1, 1)^t$, and an eigenvalue of multiplicity 2 at $\lambda_2 = 2$ with linearly independent eigenvectors $\mathbf{s}_2 = (1, 1, 0)^t$ and $\mathbf{s}_3 = (-2, 0, 1)^t$. Let

$$S_1 = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}, \quad S_2 = \{\mathbf{s}_2, \mathbf{s}_1, \mathbf{s}_3\}, \quad \text{and} \quad S_3 = \{\mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_1\}.$$

Then

$$A = S_1^{-1}D_1S_1 = S_2^{-1}D_2S_2 = S_3^{-1}D_3S_3,$$

so A is similar to D_1, D_2, and D_3.

16. (i) Let the columns of Q be denoted by the vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, which are also the rows of Q^t. Because Q is orthogonal, $(\mathbf{q}_i)^t \cdot \mathbf{q}_j$ is zero when $i \neq j$ and 1 when $i = j$. But the ij- entry of Q^tQ is $(\mathbf{q}_i)^t \cdot \mathbf{q}_j$ for each i and j so $Q^tQ = I$. Hence $Q^t = Q^{-1}$.

(ii) From part (i) we have $Q^tQ = I$, so

$$(Q\mathbf{x})^t(Q\mathbf{y}) = (\mathbf{x}^tQ^t)(Q\mathbf{y}) = \mathbf{x}^t(Q^tQ)\mathbf{y} = \mathbf{x}^t(I)\mathbf{y} = \mathbf{x}^t\mathbf{y}.$$

(iii) This follows from part (ii) with $\mathbf{x}$ replacing $\mathbf{y}$, since then

$$||Q\mathbf{x}||_2^2 = (Q\mathbf{x})^t(Q\mathbf{x}) = \mathbf{x}^t\mathbf{x} = ||\mathbf{x}||_2^2.$$

17. The matrix A has an eigenvalue of multiplicity 1 at $\lambda_1 = 3$, and an eigenvalue of multiplicity 2 at $\lambda_2 = 2$. However, $\lambda_2 = 2$ has only one linearly independent unit eigenvector, so by Theorem 9.13, A is not similar to a diagonal matrix.

18. Let the columns of Q be denoted by the vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, which are also the rows of $Q^t = Q^{-1}$. Then $Q^tQ = I$ is equivalent to having $(\mathbf{q}_i)^t \cdot \mathbf{q}_j = 0$ when $i \neq j$ and $(\mathbf{q}_i)^t \cdot \mathbf{q}_i = 1$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. But this is precisely what is required for the columns of Q to form an orthonormal set. Hence Q is an orthogonal matrix.

19. The proof of Theorem 9.13 follows by considering the form the diagonal matrix must assume. The matrix A is similar to a diagonal matrix D if and only if an invertible matrix S exists with $D = S^{-1}AS$, which is equivalent to $AS = SD$, with S invertible. Suppose that we have $AS = SD$ with the columns of S denoted $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ and the diagonal elements of D denoted $d_1, d_2, \ldots, d_n$. Then $A\mathbf{s}_i = d_i\mathbf{s}_i$ for each $i = 1, 2, \ldots, n$. Hence each d_i is an eigenvalue of A with corresponding eigenvector $\mathbf{s}_i$. The matrix S is invertible, and consequently A is similar to D, if and only if there are n linearly independent eigenvectors that can be placed in the columns of S.