1. (a) The eigenvalues and associated eigenvectors are $\lambda_1 = 2, \mathbf{v}^{(1)} = (1,0,0)^t$; $\lambda_2 = 1, \mathbf{v}^{(2)} = (0,2,1)^t$; and $\lambda_3 = -1, \mathbf{v}^{(3)} = (-1,1,1)^t$. The set is linearly independent.

   (b) The eigenvalues and associated eigenvectors are $\lambda_1 = 2, \mathbf{v}^{(1)} = (0,1,0)^t$; $\lambda_2 = 3, \mathbf{v}^{(2)} = (1,0,1)^t$; and $\lambda_3 = 1, \mathbf{v}^{(3)} = (1,0,-1)^t$. The set is linearly independent.

   (c) The eigenvalues and associated eigenvectors are $\lambda_1 = 1, \mathbf{v}^{(1)} = (0,-1,1)^t; \lambda_2 = 1 + \sqrt{2}, \mathbf{v}^{(2)} = (\sqrt{2},1,1)^t$; and $\lambda_3 = 1 - \sqrt{2}, \mathbf{v}^{(3)} = (-\sqrt{2},1,1)^t$. The set is linearly independent.

   (d) The eigenvalues and associated eigenvectors are $\lambda_1 = \lambda_2 = 2$ with $\mathbf{v}^{(1)} = (1,0,0)^t$ and $\lambda_3 = 3$ with $\mathbf{v}^{(3)} = (0,1,1)^t$. There are not three linearly independent eigenvectors.

2. (a) Eigenvalue $\lambda_1 = 1$ has multiplicity 3 and eigenvectors $\mathbf{v}^{(1)} = (-1,1,0)^t$ and $\mathbf{v}^{(2)} = (1,0,1)^t$. There are not three linearly independent eigenvectors.

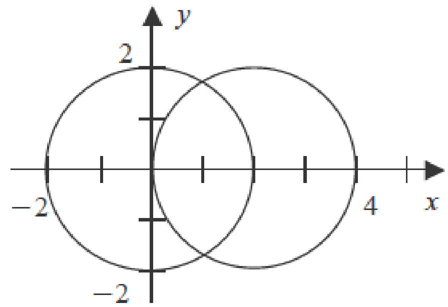
   (b) Eigenvalue $\lambda_1 = 3$ has multiplicity 2 and eigenvectors $\mathbf{v}^{(1)} = (-1,1,0)^t$ and $\mathbf{v}^{(2)} = (-1,0,1)^t$. Eigenvalue $\lambda_2 = 0$ has eigenvector $\mathbf{v}^{(3)} = (1,1,1)^t$. There are three linearly independent eigenvectors.

   (c) Eigenvalue $\lambda_1 = 4$ has eigenvector $\mathbf{v}^{(1)} = (1,1,1)^t$. Eigenvalue $\lambda_2 = 1$ has multiplicity 2 and eigenvectors $\mathbf{v}^{(2)} = (-1,1,0)^t$ and $\mathbf{v}^{(3)} = (-1,0,1)^t$. There are three linearly independent eigenvectors.

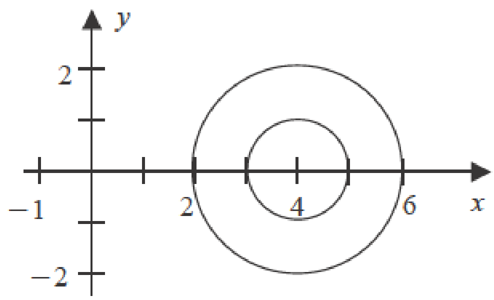
   (d) Eigenvalue $\lambda_1 = 2$ has multiplicity 2 and eigenvectors $\mathbf{v}^{(1)} = (1,0,0)^t$ and $\mathbf{v}^{(2)} = (0,-1,1)^t$. Eigenvalue $\lambda_2 = 3$ has eigenvector $\mathbf{v}^{(3)} = (1,1,0)^t$. There are three linearly independent eigenvectors.

3. The eigenvalues are within the Geršgorin circles that are shown.

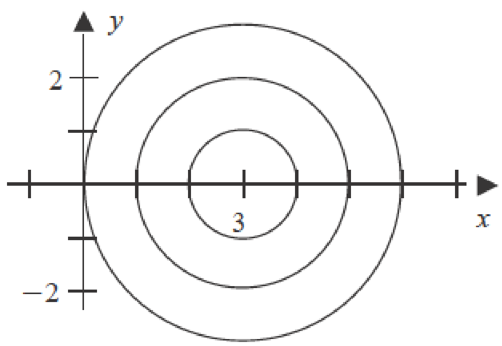
(a) The three eigenvalues are within $\{\lambda| \, |\lambda| \leq 2\} \cup \{\lambda| \, |\lambda - 2| \leq 2\}$ so $\rho(A) \leq 4$.



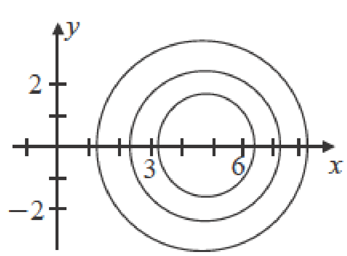
(b) The three eigenvalues are within $\{\lambda| \, |\lambda - 4| \leq 2\}$ so $\rho(A) \leq 6$.



(c) The three real eigenvalues satisfy $0 \leq \lambda \leq 6$ so $\rho(A) \leq 6$.

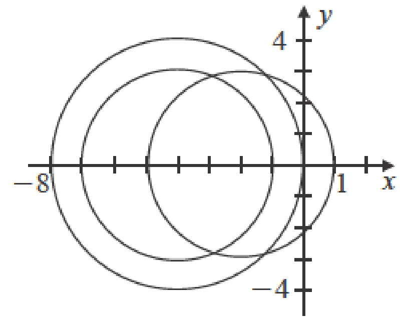


(d) The three real eigenvalues satisfy $1.25 \leq \lambda \leq 8.25$ so $1.25 \leq \rho(A) \leq 8.25$.

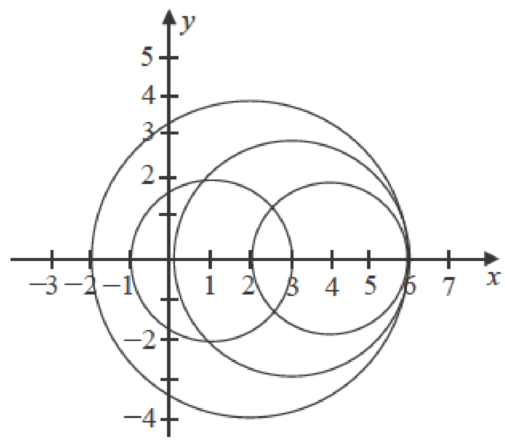


4. The eigenvalues are within the Geršgorin circles that are shown.

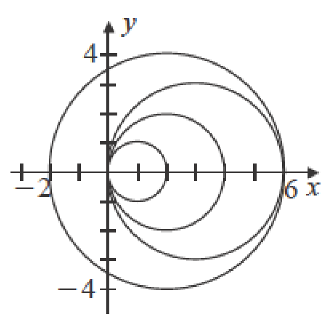
(a) The four real eigenvalues satisfy $-8 \le \lambda \le 1$, so $\rho(A) \le 8$.



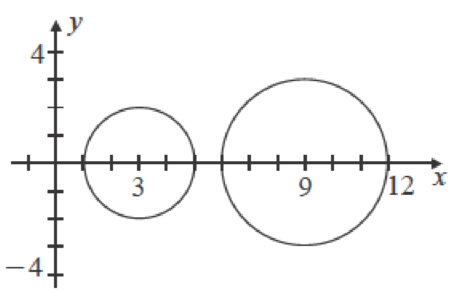
(b) The real eigenvalues satisfy $-2 \le \lambda \le 6$, so $\rho(A) \le 6$.



(c) The four real eigenvalues satisfy $-2 \le \lambda \le 6$, so $\rho(A) \le 6$.



(d) The real eigenvalues satisfy either $1 \le \lambda \le 5$ or $6 \le \lambda \le 12$, so $1 \le \rho(A) \le 12$.



5. All the matrices except (d) have 3 linearly independent eigenvectors. The matrix in part (d) has only 2 linearly independent eigenvectors. One choice for P is each case is

(a)

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

(b)

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

(c)

$$\begin{bmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

6. (a) This matrix is not factorable because it does not have 3 linearly independent eigenvectors.

(b)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}, \quad \text{and} \quad D = P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(c)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad D = P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(d)

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad D = P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

7. The vectors are linearly dependent because $-2v_1 + 7v_2 - 3v_3 = 0$.

8. The vectors are linearly independent if and only if the matrix formed by having these vectors as columns (or rows) is nonsingular, which is true if and only if the determinant of this matrix is nonzero. Since

$$\det\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2,$$

the vectors are linearly independent.

9. If $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, then for any j, with $1 \le j \le k$, we have $c_1\mathbf{v}_j^t\mathbf{v}_1 + \cdots + c_k\mathbf{v}_j^t\mathbf{v}_k = \mathbf{0}$. But orthogonality gives $c_i\mathbf{v}_j^t\mathbf{v}_i = 0$, for $i \ne j$, so $c_j\mathbf{v}_j^t\mathbf{v}_j = 0$ and since $\mathbf{v}_j^t\mathbf{v}_j \ne 0$, we must have $c_j = 0$.

10. There must be a largest subset $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$, with $j \le k$ that is linearly independent because the set with the nonzero vector $\{\mathbf{x}_1\}$ is linearly independent. Suppose that we have this largest linearly independent set and that $j < k$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \mathbf{x}_{j+1}\}$ is linearly dependent and there is a set of constants $\{c_1, c_2, \ldots, c_j\}$, not all zero, with

$$\mathbf{x}_{j+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_j\mathbf{x}_j.$$

Because these are all eigenvectors, $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for each $i = 1, 2, \ldots j+1$, so

$$A\mathbf{x}_{j+1} = \lambda_{j+1}\mathbf{x}_{j+1} = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_j\lambda_j\mathbf{x}_j.$$

But we also have

$$\lambda_{j+1}\mathbf{x}_{j+1} = c_1\lambda_{j+1}\mathbf{x}_1 + c_2\lambda_{j+1}\mathbf{x}_2 + \cdots + c_j\lambda_{j+1}\mathbf{x}_j,$$

and subtracting these equations gives

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{j+1})\mathbf{x}_1 + c_2(\lambda_2 - \lambda_{j+1})\mathbf{x}_2 + \cdots c_j(\lambda_j - \lambda_{j+1})\mathbf{x}_j.$$

But the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j\}$ was assumed to be linearly independent, so we must have

$$0 = c_1(\lambda_1 - \lambda_{j+1}) = c_2(\lambda_2 - \lambda_{j+1}) = \cdots = c_j(\lambda_j - \lambda_{j+1}).$$

Since eigenvalues are all distinct, this implies that $0 = c_1 = c_2 = \cdots = c_j$. This contradicts the original statement that $\mathbf{x}_{j+1}$ could be written as a combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j$. The only assumption that was made was that $j < k$, and this statement must be false. As a consequence, the entire set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ must be linearly independent.

11. Since $\{\mathbf{v}_i\}_{i=1}^n$ is linearly independent in $\mathbb{R}^n$, there exist numbers $c_1, \ldots, c_n$ with

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

Hence, for any k, with $1 \le k \le n$,

$$\mathbf{v}_k^t\mathbf{x} = c_1\mathbf{v}_k^t\mathbf{v}_1 + \cdots + c_n\mathbf{v}_k^t\mathbf{v}_n = c_k\mathbf{v}_k^t\mathbf{v}_k = c_k.$$

12. Not necessarily. Consider the vectors $\mathbf{x}_1 = (1,0)^t$, $\mathbf{x}_2 = (0,1)^t$, and $\mathbf{x}_3 = (1,1)^t$.

13. (a) (i) $\mathbf{0} = c_1(1,1)^t + c_2(-2,1)^t$ implies that $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. But $\det \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = 3 \neq 0$ so by Theorem 6.7 we have $c_1 = c_2 = 0$.
(ii) $\{(1,1)^t, (-3/2, 3/2)^t\}$.
(iii) $\{(\sqrt{2}/2, \sqrt{2}/2)^t, (-\sqrt{2}/2, \sqrt{2}/2)^t\}$.

(b) (i) The determinant of this matrix is $-2 \neq 0$, so $\{(1,1,0)^t, (1,0,1)^t, (0,1,1)^t\}$ is a linearly independent set.
(ii) $\{(1,1,0)^t, (1/2, -1/2, 1)^t, (-2/3, 2/3, 2/3)^t\}$
(iii) $\{(\sqrt{2}/2, \sqrt{2}/2, 0)^t, (\sqrt{6}/6, -\sqrt{6}/6, \sqrt{6}/3)^t, (-\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^t\}$

(c) (i) If $\mathbf{0} = c_1(1,1,1,1)^t + c_2(0,2,2,2)^t + c_3(1,0,0,1)^t$, then we have

$$(E_1): c_1 + c_3 = 0, \quad (E_2): c_1 + 2c_2 = 0, \quad (E_3): c_1 + 2c_2 = 0, \quad (E_4): c_1 + 2c_2 + c_3 = 0.$$

Subtracting (E_3) from (E_4) implies that $c_3 = 0$. Hence, from (E_1) we have $c_1 = 0$, and from (E_2) we have $c_2 = 0$. The vectors are linearly independent.
(ii) $\{(1,1,1,1)^t, (-3/2, 1/2, 1/2, 1/2)^t, (0, -1/3, -1/3, 2/3)^t\}$
(iii) $\{(1/2, 1/2, 1/2, 1/2)^t, (-\sqrt{3}/2, \sqrt{3}/6, \sqrt{3}/6, \sqrt{3}/6)^t, (0, -\sqrt{6}/6, -\sqrt{6}/6, \sqrt{6}/3)^t\}$

(d) (i) If A is the matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$, then $\det A = 60 \neq 0$, so the vectors are linearly independent.
(ii) $\{(2,2,3,2,3)^t, (2,-1,0,-1,0)^t, (0,0,1,0,-1)^t, (1,2,-1,0,-1)^t, (-2/7, 3/7, 2/7, -1, 2/7)^t\}$
(iii) $\{(\sqrt{30}/15, \sqrt{30}/15, \sqrt{30}/10, \sqrt{30}/15, \sqrt{30}/10)^t, (\sqrt{6}/3, -\sqrt{6}/6, 0, -\sqrt{6}/6, 0)^t,$
$(0, 0, \sqrt{2}/2, 0, -\sqrt{2}/2)^t, (\sqrt{7}/7, 2\sqrt{7}/7, -\sqrt{7}/7, 0, -\sqrt{7}/7)^t,$
$(-\sqrt{70}/35, 3\sqrt{70}/70, \sqrt{70}/35, -\sqrt{70}/10, \sqrt{70}/35)^t\}$

14. To show linear independence we will show that the matrix whose columns are the given vectors has a nonzero determinant. In case there are fewer vectors than the dimension of the space we can add a vector or vectors to complete the matrix. If the larger set is linearly independent, the original set will also be linearly independent.

(a) (i) The matrix has determinant 7.

(ii) The set $\left\{(2,-1)^t, \frac{1}{5}(1,2)^t\right\}$ is orthogonal.

(iii) The set $\left\{\frac{\sqrt{5}}{5}(2,-1)^t, \frac{\sqrt{5}}{5}(1,2)^t\right\}$ is orthonormal.

(b) (i) The matrix has determinant -2.

(ii) The set $\left\{(2,-1,1)^t, \frac{1}{2}(0,1,1)^t, \frac{2}{3}(1,1,-1)^t\right\}$ is orthogonal.

(iii) The set $\left\{\frac{\sqrt{6}}{6}(2,1-,1)^t, \frac{\sqrt{2}}{2}(0,1,1)^t, \frac{\sqrt{3}}{3}(1,1,-1)^t\right\}$ is orthonormal.

(c) (i) The matrix with the vector $(0,0,0,1)^t$ added in the last column has determinant 1.

(ii) The set $\left\{(1,1,1,1)^t, \frac{1}{4}(-3,1,1,1)^t, \frac{1}{3}(0,-2,1,1)^t\right\}$ is orthogonal.

(iii) The set $\left\{\frac{1}{2}(1,1,1,1)^t, \frac{\sqrt{3}}{6}(3,-1,-1,-1)^t, \frac{\sqrt{6}}{6}(0,-2,1,1)^t\right\}$ is orthonormal.

(d) (i) The matrix with the vector $(0,0,1,0,0)^t$ added in the last column has determinant 12.

(ii) The set $\left\{(2,2,0,2,1)^t, \frac{3}{13}(-5,8,0,-5,4)^t, \frac{1}{10}(-5,2,0,5,-4)^t, \frac{4}{7}(-1,-1,0,1,2)^t\right\}$ is orthogonal.

(iii) The set

$$\left\{\frac{\sqrt{13}}{13}(2,2,0,2,1)^t, \frac{13\sqrt{130}}{1690}(5,-8,0,5,-4)^t, \frac{\sqrt{70}}{70}(5,-2,0,-5,4)^t, -\frac{\sqrt{7}}{7}(1,1,0,-1,-2)^t\right\}$$

is orthonormal.

15. If A is a strictly diagonally dominant matrix, then in each row, the sum of the magnitudes of the off-diagonal entries in the row are less than the magnitude of the diagonal entry in that row. By Geršgorin Circle Theorem this implies that for each row the magnitude of the center of the Geršgorin circle for that row exceeds the radius so the circle does not contain the origin. Hence 0 cannot be in any Geršgorin circle and consequently cannot be an eigenvalue of A. This implies that A in nonsingular.

16. Let $(X)_k = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and define the set $(V)_k = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in the Gram-Schmidt manner as

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \text{and} \quad \mathbf{v}_k = \mathbf{x}_k - \sum_{i=1}^{k-1} \left(\frac{\mathbf{v}_i^t \mathbf{x}_k}{\mathbf{v}_i^t \mathbf{v}_i} \right) \mathbf{v}_i.$$

for each $k > 1$. We will use Mathematical Induction to show that $(V)_k$ is orthogonal for every integer k.

First note that since

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \text{and} \quad \mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1^t \mathbf{x}_2}{\mathbf{v}_1^t \mathbf{v}_1} \right) \mathbf{v}_1,$$

we have

$$\mathbf{v}_1^t \cdot \mathbf{v}_2 = \mathbf{v}_1^t \cdot \mathbf{x}_2 - \left(\frac{\mathbf{v}_1^t \mathbf{x}_2}{\mathbf{v}_1^t \mathbf{v}_1} \right) \mathbf{v}_1^t \cdot \mathbf{v}_1 = \mathbf{v}_1^t \cdot \mathbf{x}_2 - \mathbf{v}_1^t \cdot \mathbf{x}_2 = 0,$$

so $(V)_2$ is an orthogonal set.

Now assume that $(V)_j$ is orthogonal for some positive integer j, and consider $(V)_{j+1}$. Since $(V)_j$ is orthogonal the set $(V)_{j+1}$ will be orthogonal if and only if $\mathbf{v}_s^t \cdot \mathbf{v}_{j+1} = 0$ for each $s = 1, 2, \ldots j$.

For each $s = 1, 2, \ldots j$ we have

$$\begin{aligned} \mathbf{v}_s^t \cdot \mathbf{v}_{j+1}^t &= \mathbf{v}_s^t \cdot \mathbf{x}_{j+1} - \sum_{i=1}^{j} \left(\frac{\mathbf{v}_i^t \mathbf{x}_{j+1}}{\mathbf{v}_i^t \mathbf{v}_i} \right) (\mathbf{v}_s^t \cdot \mathbf{v}_i) \\ &= \mathbf{v}_s^t \cdot \mathbf{x}_{j+1} - \left(\frac{\mathbf{v}_s^t \mathbf{x}_{j+1}}{\mathbf{v}_s^t \mathbf{v}_s} \right) (\mathbf{v}_s^t \cdot \mathbf{v}_s) = \mathbf{v}_s^t \cdot \mathbf{x}_{j+1} - \mathbf{v}_s^t \cdot \mathbf{x}_{j+1}^t = 0. \end{aligned}$$

So $(V)_j$ being orthogonal implies that $(V)_{j+1}$ is also orthogonal. Mathematical Induction implies that $(V)_j$ is true for all positive integers j.

17. (a) Let μ be an eigenvalue of A. Since A is symmetric, μ is real and Theorem 9.13 gives $0 \le \mu \le 4$. The eigenvalues of $A - 4I$ are of the form $\mu - 4$. Thus,

$$\rho(A - 4I) = \max|\mu - 4| = \max(4 - \mu) = 4 - \min\mu = 4 - \lambda = |\lambda - 4|.$$

(b) The eigenvalues of $A - 4I$ are -3.618034, -2.618034, -1.381966, and -0.381966, so $\rho(A - 4I) = 3.618034$ and $\lambda = 0.381966$. An eigenvector is $(0.618034, 1, 1, 0.618034)^t$.

(c) As in part (a), $0 \le \mu \le 6$, so $|\lambda - 6| = \rho(B - 6I)$.

(d) The eigenvalues of $B - 6I$ are -5.2360673, -4, -2, and -0.76393202, so $\rho(B - 6I) = 5.2360673$ and $\lambda = 0.7639327$. An eigenvector is $(0.61803395, 1, 1, 0.6180395)^t$.