1. $S_3(x) = 2\sin x - \sin 2x$

2. $S_2(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

3. $S_3(x) = 3.676078 - 3.676078\cos x + 1.470431\cos 2x - 0.7352156\cos 3x + 3.676078\sin x - 2.940862\sin 2x$

4. The general trigonometric least-squares polynomial is

$$S_n(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \frac{(-1)^n e^{\pi} + (-1)^{n+1} e^{-\pi}}{\pi(n^2+1)} \cos nx + \frac{1}{\pi} \sum_{k=1}^{n-1} \left[\frac{(-1)^k e^{\pi} + (-1)^{k+1} e^{-\pi}}{k^2+1} \right] (\cos kx - k \sin kx).$$

5. The general trigonometric least-squares polynomial is

$$S_n(x) = \sum_{k=1}^{n-1} \frac{2}{k\pi} \left(1 - (-1)^k\right) \sin kx.$$

6. $S_n(x) = \dfrac{1}{2} + \dfrac{1}{\pi} \displaystyle\sum_{k=1}^{n-1} \dfrac{1-(-1)^k}{k} \sin kx$

7. The trigonometric least-squares polynomials are:

   (a) $S_2(x) = \cos 2x$

   (b) $S_2(x) = 0$

   (c) $S_3(x) = 1.566453 + 0.5886815\cos x - 0.2700642\cos 2x + 0.2175679\cos 3x + 0.8341640\sin x - 0.3097866\sin 2x$

   (d) $S_3(x) = -2.046326 + 3.883872\cos x - 2.320482\cos 2x + 0.7310818\cos 3x$

8. (a) $E(S_2) = 0$

   (b) $E(S_2) = 4$

   (c) $E(S_3) = 0.8259814$

   (d) $E(S_3) = 1.936668$

9. The trigonometric least-squares polynomial is

$$S_3(x) = -0.4968929 + 0.2391965\cos x + 1.515393\cos 2x + 0.2391965\cos 3x - 1.150649\sin x,$$

with error $E(S_3) = 7.271197$.

10. The trigonometric least-squares polynomial is

$$S_3(x) = 0.06201467 - 0.8600803\cos x + 2.549330\cos 2x - 0.6409933\cos 3x \\ - 0.8321197\sin x - 0.6695062\sin 2x,$$

with error 107.913.

The approximation in Exercise 10 is better because, in this case,

$$\sum_{j=0}^{10} (f(\xi_j) - S_3(\xi_j))^2 = 397.3678,$$

whereas the approximation in Exercise 9 has

$$\sum_{j=0}^{10} (f(\xi_j) - S_3(\xi_j))^2 = 569.3589.$$

11. The trigonometric least-squares polynomials and their errors are

(a)

$$S_3(x) = -0.08676065 - 1.446416\cos\pi(x-3) - 1.617554\cos 2\pi(x-3) + 3.980729\cos 3\pi(x-3) \\ - 2.154320\sin\pi(x-3) + 3.907451\sin 2\pi(x-3)$$

with $E(S_3) = 210.90453$.

(b)

$$S_4(x) = -0.0867607 - 1.446416\cos\pi(x-3) - 1.617554\cos 2\pi(x-3) + 3.980729\cos 3\pi(x-3) \\ - 2.354088\cos 4\pi(x-3) - 2.154320\sin\pi(x-3) + 3.907451\sin 2\pi(x-3) \\ - 1.166181\sin 3\pi(x-3)$$

with $E(S_4) = 169.4943$.

12. (a) The trigonometric least-squares polynomial is

$$S_4(x) = 0.2772149 - 0.1180378\cos(2x-1)\pi + 0.05649078\cos 2(2x-1)\pi \\ - 0.08807404\cos 3(2x-1)\pi + 0.04715799\cos 4(2x-1)\pi + 0.1376064\sin(2x-1)\pi \\ - 0.001375524\sin 2(2x-1)\pi + 0.02118788\sin 3(2x-1)\pi.$$

(b) $\int_0^1 S_4(x)dx = 0.27721486$

(c) $\int_0^1 x^2 \sin x dx = 0.2232443$

13. Let $f(-x) = -f(x)$. The integral $\int_{-a}^{0} f(x)\,dx$ under the change of variable $t = -x$ transforms to

$$-\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = -\int_0^a f(t)\,dt = -\int_0^a f(x)\,dx.$$

Thus,

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$$

14. Let $f(-x) = f(x)$. The integral $\int_{-a}^{0} f(x)\,dx$ under the change of variable $t = -x$ transforms to

$$-\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx.$$

Thus

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

15. The following integrations establish the orthogonality.

$$\int_{-\pi}^{\pi} [\phi_0(x)]^2\,dx = \frac{1}{2}\int_{-\pi}^{\pi} dx = \pi,$$

$$\int_{-\pi}^{\pi} [\phi_k(x)]^2\,dx = \int_{-\pi}^{\pi} (\cos kx)^2\,dx = \int_{-\pi}^{\pi} \left[\frac{1}{2} + \frac{1}{2}\cos 2kx\right]\,dx = \pi + \left[\frac{1}{4k}\sin 2kx\right]_{-\pi}^{\pi} = \pi,$$

$$\int_{-\pi}^{\pi} [\phi_{n+k}(x)]^2\,dx = \int_{-\pi}^{\pi} (\sin kx)^2\,dx = \int_{-\pi}^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos 2kx\right]\,dx = \pi - \left[\frac{1}{4k}\sin 2kx\right]_{-\pi}^{\pi} = \pi,$$

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_0(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos kx\,dx = \left[\frac{1}{2k}\sin kx\right]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_0(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin kx\,dx = \left[\frac{-1}{2k}\cos kx\right]_{-\pi}^{\pi} = \frac{-1}{2k}[\cos k\pi - \cos(-k\pi)] = 0,$$

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_j(x)\,dx = \int_{-\pi}^{\pi} \cos kx \cos jx dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(k+j)x + \cos(k-j)x]\,dx = 0,$$

$$\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_{n+j}(x)\,dx = \int_{-\pi}^{\pi} \sin kx \sin jx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(k-j)x - \cos(k+j)x]\,dx = 0,$$

and

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_{n+j}(x)\,dx = \int_{-\pi}^{\pi} \cos kx \sin jx\;dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(k+j)x - \sin(k-j)x]\,dx = 0.$$

16. The Fourier Series for $f(x) = |x|$ is

$$S(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k - 1}{k^2}\cos kx.$$

Assuming $f(0) = S(0)$ gives

$$0 = S(0) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k - 1}{k^2},$$

but

$$\sum_{k=1}^{\infty}\frac{(-1)^k - 1}{k^2} = -\frac{2}{1^2} + \frac{0}{2^2} - \frac{2}{3^2} + \frac{0}{4^2} + \cdots = -\sum_{k=0}^{\infty}\frac{2}{(2k+1)^2}.$$

Thus

$$0 = \frac{\pi}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{2}{(2k+1)^2},$$

from which

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

follows.

17. The steps are nearly identical to those for determining the constants b_k except for the additional constant term a_0 in the cosine series. In this case

$$0 = \frac{\partial E}{\partial a_0} = 2\sum_{j=0}^{2m-1}[y_j - S_n(x_j)](-1/2) = \sum_{j=0}^{2m-1} y_j - \sum_{j=0}^{2m-1}\left(\frac{a_0}{2} + a_n\cos nx_j + \sum_{k=1}^{n-1}(a_k\cos kx_j + b_k\sin kx_j)\right),$$

The orthogonality implies that only the constant term remains in the second sum. So

$$0 = \sum_{j=0}^{2m-1} y_j - \frac{a_0}{2}(2m) \quad \text{which implies that} \quad a_0 = \frac{1}{m}\sum_{j=0}^{2m-1} y_j.$$