1. The linear least-squares approximations are:

   (a) $P_1(x) = 1.833333 + 4x$
   (b) $P_1(x) = -1.600003 + 3.600003x$
   (c) $P_1(x) = 1.140981 - 0.2958375x$
   (d) $P_1(x) = 0.1945267 + 3.000001x$
   (e) $P_1(x) = 0.6109245 + 0.09167105x$
   (f) $P_1(x) = -1.861455 + 1.666667x$

2. The linear least-squares approximations on $[-1, 1]$ are:

   (a) $P_1(x) = 3.333333 - 2x$
   (b) $P_1(x) = 0.6000025x$
   (c) $P_1(x) = 0.5493063 - 0.2958375x$
   (d) $P_1(x) = 1.175201 + 1.103639x$
   (e) $P_1(x) = 0.4207355 + 0.4353975x$
   (f) $P_1(x) = 0.6479184 + 0.5281226x$

3. The least squares approximations of degree two are:

   (a) $P_2(x) = 2 + 3x + x^2 \equiv f(x)$
   (b) $P_2(x) = 0.4000163 - 2.400054x + 3.000028x^2$
   (c) $P_2(x) = 1.723551 - 0.9313682x + 0.1588827x^2$
   (d) $P_2(x) = 1.167179 + 0.08204442x + 1.458979x^2$
   (e) $P_2(x) = 0.4880058 + 0.8291830x - 0.7375119x^2$
   (f) $P_2(x) = -0.9089523 + 0.6275723x + 0.2597736x^2$

4. The least squares approximation of degree two on $[-1, 1]$ are:

   (a) $P_2(x) = 3 - 2x + 1.000009x^2$
   (b) $P_2(x) = 0.6000025x$
   (c) $P_2(x) = 0.4963454 - 0.2958375x + 0.1588827x^2$
   (d) $P_2(x) = 0.9962918 + 1.103639x + 0.5367282x^2$
   (e) $P_2(x) = 0.4982798 + 0.4353975x - 0.2326330x^2$
   (f) $P_2(x) = 0.6947898 + 0.5281226x - 0.1406141x^2$

5. The errors E for the least squares approximations in Exercise 3 are:

(a) 0.3427×10^{-9}
(b) 0.0457142
(c) 0.000358354
(d) 0.0106445
(e) 0.0000134621
(f) 0.0000967795

6. The errors for the approximations in Exercise 4 are:

(a) 0
(b) 0.0457206
(c) 0.00035851
(d) 0.0014082
(e) 0.00575753
(f) 0.00011949

7. The Gram-Schmidt process produces the following collections of polynomials:

(a) $\phi_0(x) = 1, \phi_1(x) = x - 0.5, \quad \phi_2(x) = x^2 - x + \frac{1}{6}, \quad$ and $\quad \phi_3(x) = x^3 - 1.5x^2 + 0.6x - 0.05$
(b) $\phi_0(x) = 1, \phi_1(x) = x - 1, \quad \phi_2(x) = x^2 - 2x + \frac{2}{3}, \quad$ and $\quad \phi_3(x) = x^3 - 3x^2 + \frac{12}{5}x - \frac{2}{5}$
(c) $\phi_0(x) = 1, \phi_1(x) = x - 2, \quad \phi_2(x) = x^2 - 4x + \frac{11}{3}, \quad$ and $\quad \phi_3(x) = x^3 - 6x^2 + 11.4x - 6.8$

8. The Gram-Schmidt process produces the following collections of polynomials.

(a) $3.833333\phi_0(x) + 4.000000\phi_1(x)$
(b) $2\phi_0(x) + 3.6\phi_1(x)$
(c) $0.5493061\phi_0(x) - 0.2958369\phi_1(x)$
(d) $3.194528\phi_0(x) + 3\phi_1(x)$
(e) $0.6567600\phi_0(x) + 0.09167105\phi_1(x)$
(f) $1.471878\phi_0(x) + 1.666667\phi_1(x)$

9. The least-squares polynomials of degree three are:

   (a) $P_3(x) = 3.833333\phi_0(x) + 4.000000\phi_1(x) + 0.9999998\phi_2(x)$
   (b) $P_3(x) = 2\phi_0(x) + 3.6\phi_1(x) + 3\phi_2(x) + \phi_3(x)$
   (c) $P_3(x) = 0.5493061\phi_0(x) - 0.2958369\phi_1(x) + 0.1588785\phi_2(x) - 0.08524470\phi_3(x)$
   (d) $P_3(x) = 3.194528\phi_0(x) + 3\phi_1(x) + 1.458960\phi_2(x) + 0.4787959\phi_3(x)$
   (e) $P_3(x) = 0.6567600\phi_0(x) + 0.09167105\phi_1(x) - 0.7375118\phi_2(x) - 0.1876952\phi_3(x)$
   (f) $P_3(x) = 1.471878\phi_0(x) + 1.666667\phi_1(x) + 0.2597705\phi_2(x) - 0.04559611\phi_3(x)$

10. The least-squares polynomials of degree two are:

   (a) $P_2(x) = 3.833333\phi_0(x) + 4\phi_1(x) + 0.9999998\phi_2(x)$
   (b) $P_2(x) = 2\phi_0(x) + 3.6\phi_1(x) + 3\phi_2(x)$
   (c) $P_2(x) = 0.5493061\phi_0(x) - 0.2958369\phi_1(x) + 0.1588785\phi_2(x)$
   (d) $P_2(x) = 3.194528\phi_0(x) + 3\phi_1(x) + 1.458960\phi_2(x)$
   (e) $P_2(x) = 0.6567600\phi_0(x) + 0.09167105\phi_1(x) - 0.73751218\phi_2(x)$
   (f) $P_2(x) = 1.471878\phi_0(x) + 1.666667\phi_1(x) + 0.2597705\phi_2(x)$

11. The Laguerre polynomials are $L_1(x) = x-1$, $L_2(x) = x^2-4x+2$ and $L_3(x) = x^3-9x^2+18x-6$.

12. The least-squares polynomials of degrees one, two, and three are:

   (a) $2L_0(x) + 4L_1(x) + L_2(x)$
   (b) $\frac{1}{2}L_0(x) - \frac{1}{4}L_1(x) + \frac{1}{16}L_2(x) - \frac{1}{96}L_3(x)$
   (c) $6L_0(x) + 18L_1(x) + 9L_2(x) + L_3(x)$
   (d) $\frac{1}{3}L_0(x) - \frac{2}{9}L_1(x) + \frac{2}{27}L_2(x) - \frac{4}{243}L_3(x)$

13. Let $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ be a linearly independent set of polynomials in $\prod_n$. For each $i = 0, 1, \ldots, n$, let $\phi_i(x) = \sum_{k=0}^{n} b_{ki} x^k$. Let $Q(x) = \sum_{k=0}^{n} a_k x^k \in \prod_n$. We want to find constants $c_0, \ldots, c_n$ so that

$$Q(x) = \sum_{i=0}^{n} c_i \phi_i(x).$$

This equation becomes

$$\sum_{k=0}^{n} a_k x^k = \sum_{i=0}^{n} c_i \left(\sum_{k=0}^{n} b_{ki} x^k \right)$$

so we have both

$$\sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} \left(\sum_{i=0}^{n} c_i b_{ki} \right) x^k, \quad \text{and} \quad \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} \left(\sum_{i=0}^{n} b_{ki} c_i \right) x^k.$$

But $\{1, x, \ldots, x^n\}$ is linearly independent, so, for each $k = 0, \ldots, n$, we have

$$\sum_{i=0}^{n} b_{ki} c_i = a_k,$$

which expands to the linear system

$$\begin{bmatrix} b_{01} & b_{02} & \cdots & b_{0n} \\ b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \\ \\ \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

This linear system must have a unique solution $\{c_0, c_1, \ldots, c_n\}$, or else there is a nontrivial set of constants $\{c_0', c_1', \ldots, c_n'\}$, for which

$$\begin{bmatrix} b_{01} & \cdots & b_{0n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0' \\ \vdots \\ c_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus

$$c_0' \phi_0(x) + c_1' \phi_1(x) + \ldots + c_n' \phi_n(x) = \sum_{k=0}^{n} 0 x^k = 0,$$

which contradicts the linear independence of the set $\{\phi_0, \ldots, \phi_n\}$. Thus, there is a unique set of constants $\{c_0, \ldots, c_n\}$, for which

$$Q(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + \ldots + c_n \phi_n(x).$$

14. If $\sum_{i=0}^{n} c_i \phi_i(x) = 0$, for all $a \le x \le b$, then

$$\int_a^b \left(\sum_{i=0}^{n} c_i \phi_i(x) \right) \phi_j(x) w(x) \, dx = 0, \quad \text{for each } j = 0, 1, \ldots, n.$$

Thus, $c_j = 0$, for each $j = 0, 1, \ldots, n$.

15. The normal equations are

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx, \quad \text{for each} \;\; j = 0, 1, \ldots, n.$$

Let

$$b_{jk} = \int_a^b x^{j+k} dx, \quad \text{for each} \;\; j = 0, \ldots, n, \quad \text{and} \quad k = 0, \ldots, n,$$

and let $B = (b_{jk})$. Further, let

$$\mathbf{a} = (a_0, \ldots, a_n)^t \quad \text{and} \quad \mathbf{g} = \left(\int_a^b f(x) dx, \ldots, \int_a^b x^n f(x)\, dx \right)^t.$$

Then the normal equations produce the linear system $B\mathbf{a} = \mathbf{g}$.

To show that the normal equations have a unique solution, it suffices to show that if $f \equiv 0$ then $\mathbf{a} = \mathbf{0}$. If $f \equiv 0$, then

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = 0, \quad \text{for} \;\; j = 0, \ldots, n, \quad \text{and} \quad \sum_{k=0}^{n} a_j a_k \int_a^b x^{j+k} dx = 0, \quad \text{for} \;\; j = 0, \ldots, n,$$

and summing over j gives

$$\sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \int_a^b x^{j+k} dx = 0.$$

Thus

$$\int_a^b \sum_{j=0}^{n} \sum_{k=0}^{n} a_j x^j a_k x^k dx = 0 \quad \text{and} \quad \int_a^b \left(\sum_{j=0}^{n} a_j x^j \right)^2 dx = 0.$$

Define $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then $\int_a^b [P(x)]^2\, dx = 0$ and $P(x) \equiv 0$. This implies that $a_0 = a_1 = \cdots = a_n = 0$, so $\mathbf{a} = \mathbf{0}$. Hence, the matrix B is nonsingular, and the normal equations have a unique solution.