1. (a) $(0.18, 0.13)^t$
   (b) $(0.19, 0.10)^t$
   (c) Gaussian elimination gives the best answer since $\mathbf{v}^{(2)} = (0, 0)^t$ in the conjugate gradient method.
   (d) $(0.13, 0.21)^t$. There is no improvement, although $\mathbf{v}^{(2)} \neq \mathbf{0}$.

2. (a) $(1.0, 1.0)^t$
   (b) $(1.0, 1.0)^t$
   (c) Both answers are the same. However, more operations are required in the conjugate gradient method.
   (d) $(1.1, 1.0)^t$. The answer is not as good due to rounding error.

3. (a) $(1.00, -1.00, 1.00)^t$
   (b) $(0.827, 0.0453, -0.0357)^t$
   (c) Partial pivoting and scaled partial pivoting also give $(1.00, -1.00, 1.00)^t$.
   (d) $(0.776, 0.238, -0.185)^t$.

   The residual from (3b) is $(-0.0004, -0.0038, 0.0037)^t$, and the residual from part (3d) is $(0.0022, -0.0038, 0.0024)^t$. There does not appear to be much improvement, if any. Rounding error is more prevalent because of the increase in the number of matrix multiplications.

4. (a) $(0.9999999997, -1, 1)^t$
   (b) $(0.9999991959, -1.000066419, 0.9999996693)^t$;
   The residual is $\left(0.11236 \times 10^{-5}, 0.6242 \times 10^{-6}, 0.4387 \times 10^{-6}\right)^t$
   (c) Partial pivoting gives the same answer as in part (a).
   (d) $(1.000000364, -0.999999391, 1.000000888)^t$.

   The residual is $\left(-0.10001 \times 10^{-5}, -0.63087 \times 10^{-6}, -0.4691 \times 10^{-6}\right)^t$.

   There does not seem to be an improvement in this preconditioning method.

5. Two steps of the Conjugate Gradient method with $C = C^{-1} = I$ give the following:
   (a) $\mathbf{x}^{(2)} = (0.1535933456, -0.1697932117, 0.5901172091)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 0.221$.
   (b) $\mathbf{x}^{(2)} = (0.9993129510, 0.9642734456, 0.7784266575)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 0.144$.
   (c) $\mathbf{x}^{(2)} = (-0.7290954114, 2.515782452, -0.6788904058, -2.331943982)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 2.2$.
   (d) $\mathbf{x}^{(2)} = (-0.7071108901, -0.0954748881, -0.3441074093, 0.5256091497)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 0.39$.
   (e) $\mathbf{x}^{(2)} = (0.5335968381, 0.9367588935, 1.339920949, 1.743083004, 1.743083004)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 1.3$.
   (f) $\mathbf{x}^{(2)} = (1.022375671, 1.686451893, 1.022375671, 2.060919568, 0.8310997764, 2.060919568)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 1.13$.

6. Two steps of the Conjugate Gradient method with $C^{-1} = D^{-1/2}$ give the following:

   (a) $\mathbf{x}^{(2)} = (0.1012813293, -0.2095507352, 0.0701217891)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 0.145$

   (b) $\mathbf{x}^{(2)} = (0.9993129510, 0.9642734455, 0.7784266577)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 0.144$

   (c) $\mathbf{x}^{(2)} = (-0.3365802625, 2.129693189, -0.7600395580, 2.703196814)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 2.35$

   (d) $\mathbf{x}^{(2)} = (0.5927721564, -0.3791968233, -0.02649943827, 0.0197727283)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 0.146$

   (e) $\mathbf{x}^{(2)} = (0.4414248576, 0.8089276500, 1.463760200, 1.730537721, 1.895808600)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 1.06$

   (f) $\mathbf{x}^{(2)} = (1.022375670, 1.686451892, 1.022375670, 2.060919568, 0.8310997753, 2.060919568)^t$ and $\|\mathbf{r}^{(2)}\|_\infty = 1.13$

7. The Conjugate Gradient method with $C = C^{-1} = I$ gives the following:

   (a) $\mathbf{x}^{(3)} = (0.06185567013, -0.1958762887, 0.6185567010)^t$ and $\|\mathbf{r}^{(3)}\|_\infty = 0.4 \times 10^{-9}$.

   (b) $\mathbf{x}^{(3)} = (0.9957894738, 0.9578947369, 0.7915789474)^t$ and $\|\mathbf{r}^{(3)}\|_\infty = 0.1 \times 10^{-9}$.

   (c) $\mathbf{x}^{(4)} = (-0.7976470579, 2.795294120, -0.2588235305, -2.251764706)^t$ and $\|\mathbf{r}^{(4)}\|_\infty = 0.39 \times 10^{-7}$.

   (d) $\mathbf{x}^{(4)} = (-0.7534246575, 0.04109589039, -0.2808219179, 0.6917808219)^t$ and $\|\mathbf{r}^{(4)}\|_\infty = 0.11 \times 10^{-9}$.

   (e) $\mathbf{x}^{(5)} = (0.4516129032, 0.7096774197, 1.677419355, 1.741935483, 1.806451613)^t$ and $\|\mathbf{r}^{(5)}\|_\infty = 0.2 \times 10^{-9}$.

   (f) $\mathbf{x}^{(4)} = (1.000000000, 2.000000000, 1.000000000, 2.000000000, 0.9999999997, 2.000000000)^t$ and $\|\mathbf{r}^{(4)}\|_\infty = 0.44 \times 10^{-9}$.

8. The Conjugate Gradient method with $C^{-1} = D^{-1/2}$ gives the following:

   (a) $\mathbf{x}^{(3)} = (0.06185567019, -0.1958762885, 0.6185567006)^t$ and $\|\mathbf{r}^{(3)}\|_\infty = 0.12 \times 10^{-8}$

   (b) $\mathbf{x}^{(3)} = (0.9957894739, 0.9578947368, 0.7915789475)^t$ and $\|\mathbf{r}^{(3)}\|_\infty = 0.19 \times 10^{-8}$

   (c) $\mathbf{x}^{(4)} = (-0.7976470596, 2.795294118, -0.2588235287, -2.251764706)^t$ and $\|\mathbf{r}^{(4)}\|_\infty = 0.7 \times 10^{-8}$

   (d) $\mathbf{x}^{(4)} = (0.6164383560, -0.3972602742, 0.04794520550, -0.02054794525)^t$ and $\|\mathbf{r}^{(4)}\|_\infty = 0.76 \times 10^{-9}$

   (e) $\mathbf{x}^{(5)} = (0.4516129026, 0.7096774190, 1.677419356, 1.741935484, 1.806451615)^t$ and $\|\mathbf{r}^{(5)}\|_\infty = 0.61 \times 10^{-10}$

   (f) $\mathbf{x}^{(4)} = (0.9999999992, 1.999999999, 0.9999999992, 2.000000000, 0.9999999989, 2.000000000)^t$ and $\|\mathbf{r}^{(4)}\|_\infty = 0.11 \times 10^{-9}$

9. Approximations to within 10^{-5} in the l_{infty} are shown in the tables.

(a)	Jacobi 49 iterations	Gauss-Seidel 28 iterations	SOR ($\omega = 1.3$) 13 iterations	Conjugate Gradient 9 iterations
x_1	0.93406183	0.93406917	0.93407584	0.93407713
x_2	0.97473885	0.97475285	0.97476180	0.97476363
x_3	1.10688692	1.10690302	1.10691093	1.10691243
x_4	1.42346150	1.42347226	1.42347591	1.42347699
x_5	0.85931331	0.85932730	0.85933633	0.85933790
x_6	0.80688119	0.80690725	0.80691961	0.80692197
x_7	0.85367746	0.85370564	0.85371536	0.85372011
x_8	1.10688692	1.10690579	1.10691075	1.10691250
x_9	0.87672774	0.87674384	0.87675177	0.87675250
x_{10}	0.80424512	0.80427330	0.80428301	0.80428524
x_{11}	0.80688119	0.80691173	0.80691989	0.80692252
x_{12}	0.97473885	0.97475850	0.97476265	0.97476392
x_{13}	0.93003466	0.93004542	0.93004899	0.93004987
x_{14}	0.87672774	0.87674661	0.87675155	0.87675298
x_{15}	0.85931331	0.85933296	0.85933709	0.85933979
x_{16}	0.93406183	0.93407462	0.93407672	0.93407768

(b)	Jacobi 60 iterations	Gauss-Seidel 35 iterations	SOR ($\omega = 1.2$) 23 iterations	Conjugate Gradient 11 iterations
x_1	0.39668038	0.39668651	0.39668915	0.39669775
x_2	0.07175540	0.07176830	0.07177348	0.07178516
x_3	−0.23080396	−0.23078609	−0.23077981	−0.23076923
x_4	0.24549277	0.24550989	0.24551535	0.24552253
x_5	0.83405412	0.83406516	0.83406823	0.83407148
x_6	0.51497606	0.51498897	0.51499414	0.51500583
x_7	0.12116003	0.12118683	0.12119625	0.12121212
x_8	−0.24044414	−0.24040991	−0.24039898	−0.24038462
x_9	0.37873579	0.37876891	0.37877812	0.37878788
x_{10}	1.09073364	1.09075392	1.09075899	1.09076341
x_{11}	0.54207872	0.54209658	0.54210286	0.54211344
x_{12}	0.13838259	0.13841682	0.13842774	0.13844211
x_{13}	−0.23083868	−0.23079452	−0.23078224	−0.23076923
x_{14}	0.41919067	0.41923122	0.41924136	0.41925019
x_{15}	1.15015953	1.15018477	1.15019025	1.15019425
x_{16}	0.51497606	0.51499318	0.51499864	0.51500583
x_{17}	0.12116003	0.12119315	0.12120236	0.12121212
x_{18}	−0.24044414	−0.24040359	−0.24039345	−0.24038462
x_{19}	0.37873579	0.37877365	0.37878188	0.37878788
x_{20}	1.09073364	1.09075629	1.09076069	1.09076341
x_{21}	0.39668038	0.39669142	0.39669449	0.39669775
x_{22}	0.07175540	0.07177567	0.07178074	0.07178516
x_{23}	−0.23080396	−0.23077872	−0.23077323	−0.23076923
x_{24}	0.24549277	0.24551542	0.24551982	0.24552253
x_{25}	0.83405412	0.83406793	0.83407025	0.83407148

(c)	Jacobi 15 iterations	Gauss-Seidel 9 iterations	SOR ($\omega = 1.1$) 8 iterations	Conjugate Gradient 8 iterations
x_1	−3.07611424	−3.07611739	−3.07611796	−3.07611794
x_2	−1.65223176	−1.65223563	−1.65223579	−1.65223582
x_3	−0.53282391	−0.53282528	−0.53282531	−0.53282528
x_4	−0.04471548	−0.04471608	−0.04471609	−0.04471604
x_5	0.17509673	0.17509661	0.17509661	0.17509661
x_6	0.29568226	0.29568223	0.29568223	0.29568218
x_7	0.37309012	0.37309011	0.37309011	0.37309011
x_8	0.42757934	0.42757934	0.42757934	0.42757927
x_9	0.46817927	0.46817927	0.46817927	0.46817927
x_{10}	0.49964748	0.49964748	0.49964748	0.49964748
x_{11}	0.52477026	0.52477026	0.52477026	0.52477027
x_{12}	0.54529835	0.54529835	0.54529835	0.54529836
x_{13}	0.56239007	0.56239007	0.56239007	0.56239009
x_{14}	0.57684345	0.57684345	0.57684345	0.57684347
x_{15}	0.58922662	0.58922662	0.58922662	0.58922664
x_{16}	0.59995522	0.59995522	0.59995522	0.59995523
x_{17}	0.60934045	0.60934045	0.60934045	0.60934045
x_{18}	0.61761997	0.61761997	0.61761997	0.61761998
x_{19}	0.62497846	0.62497846	0.62497846	0.62497847
x_{20}	0.63156161	0.63156161	0.63156161	0.63156161
x_{21}	0.63748588	0.63748588	0.63748588	0.63748588
x_{22}	0.64284553	0.64284553	0.64284553	0.64284553
x_{23}	0.64771764	0.64771764	0.64771764	0.64771764
x_{24}	0.65216585	0.65216585	0.65216585	0.65216585
x_{25}	0.65624320	0.65624320	0.65624320	0.65624320
x_{26}	0.65999423	0.65999423	0.65999423	0.65999422
x_{27}	0.66345660	0.66345660	0.66345660	0.66345660
x_{28}	0.66666242	0.66666242	0.66666242	0.66666242
x_{29}	0.66963919	0.66963919	0.66963919	0.66963919
x_{30}	0.67241061	0.67241061	0.67241061	0.67241060
x_{31}	0.67499722	0.67499722	0.67499722	0.67499721
x_{32}	0.67741692	0.67741692	0.67741691	0.67741691
x_{33}	0.67968535	0.67968535	0.67968535	0.67968535
x_{34}	0.68181628	0.68181628	0.68181628	0.68181628
x_{35}	0.68382184	0.68382184	0.68382184	0.68382184
x_{36}	0.68571278	0.68571278	0.68571278	0.68571278
x_{37}	0.68749864	0.68749864	0.68749864	0.68749864
x_{38}	0.68918652	0.68918652	0.68918652	0.68918652
x_{39}	0.69067718	0.69067718	0.69067718	0.69067717
x_{40}	0.68363346	0.68363346	0.68363346	0.68363349

10. $n = 10$: The solution vector is

$(0.90909091, 0.81818182, 0.72727273, 0.63636364, 0.54545455, 0.45454545, 0.36363636, 0.27272727, 0.18181818, 0.09090909)^t$,

using 10 iterations with $\|\mathbf{r}^{(10)}\|_\infty = 0$.

$n = 50$: The solution vector is

$(0.98039216, 0.96078432, 0.94117648, 0.92156863, 0.90196079, 0.88235295, 0.86274511, 0.84313727, 0.82352943, 0.80392158, 0.78431374, 0.76470590, 0.74509806, 0.72549021, 0.70588237, 0.68627453, 0.66666668, 0.64705884, 0.62745100, 0.60784315, 0.58823531, 0.56862747, 0.54901962, 0.52941178, 0.50980394, 0.49019609, 0.47058825, 0.45098041, 0.43137256, 0.41176472, 0.39215688, 0.37254903, 0.35294119, 0.33333335, 0.31372550, 0.29411766, 0.27450981, 0.25490197, 0.23529413, 0.21568628, 0.19607844, 0.17647060, 0.15686275, 0.13725491, 0.11764706, 0.09803922, 0.07843138, 0.05882353, 0.03921569, 0.01960784)^t$,

using 50 iterations with a tolerance 1.00×10^{-3} in l_∞ and $\|\mathbf{r}^{(50)}\|_\infty = 0$.

$n = 100$: The solution vector is

$(0.99009901, 0.98019803, 0.97029704, 0.96039606, 0.95049507, 0.94059409, 0.93069310, 0.92079212, 0.91089113, 0.90099014, 0.89108916, 0.88118817, 0.87128718, 0.86138620, 0.85148521, 0.84158422, 0.83168323, 0.82178225, 0.81188126, 0.80198027, 0.79207928, 0.78217830, 0.77227731, 0.76237632, 0.75247533, 0.74257434, 0.73267335, 0.72277237, 0.71287138, 0.70297039, 0.69306940, 0.68316841, 0.67326742, 0.66336643, 0.65346544, 0.64356445, 0.63366347, 0.62376248, 0.61386149, 0.60396050, 0.59405951, 0.58415852, 0.57425753, 0.56435654, 0.55445555, 0.54455456, 0.53465357, 0.52475258, 0.51485159, 0.50495059, 0.49504960, 0.48514861, 0.47524762, 0.46534663, 0.45544564, 0.44554465, 0.43564366, 0.42574267, 0.41584168, 0.40594068, 0.39603969, 0.38613870, 0.37623771, 0.36633672, 0.35643573, 0.34653474, 0.33663374, 0.32673275, 0.31683176, 0.30693077, 0.29702978, 0.28712879, 0.27722779, 0.26732680, 0.25742581, 0.24752482, 0.23762383, 0.22772283, 0.21782184, 0.20792085, 0.19801986, 0.18811886, 0.17821787, 0.16831688, 0.15841589, 0.14851489, 0.13861390, 0.12871291, 0.11881192, 0.10891092, 0.09900993, 0.08910894, 0.07920794, 0.06930695, 0.05940596, 0.04950497, 0.03960397, 0.02970298, 0.01980199, 0.00990099)^t$,

using 100 iterations with a tolerance 1.00×10^{-3} in l_∞ and $\|\mathbf{r}^{(100)}\|_\infty = 0$.

11. The Conjugate Gradient method gives the results in the following tables.

(a)

Solution	Residual
2.55613420	0.00668246
4.09171393	−0.00533953
4.60840390	−0.01739814
3.64309950	−0.03171624
5.13950533	0.01308093
7.19697808	−0.02081095
7.68140405	−0.04593118
5.93227784	0.01692180
5.81798997	0.04414047
5.85447806	0.03319707
5.94202521	−0.00099947
4.42152959	−0.00072826
3.32211695	0.02363822
4.49411604	0.00982052
4.80968966	0.00846967
3.81108707	−0.01312902

This converges in 6 iterations with tolerance 5.00×10^{-2} in the l_∞ norm and $\|\mathbf{r}^{(6)}\|_\infty =$ 0.046.

(b)

Solution	Residual
2.55613420	0.00668246
4.09171393	−0.00533953
4.60840390	−0.01739814
3.64309950	−0.03171624
5.13950533	0.01308093
7.19697808	−0.02081095
7.68140405	−0.04593118
5.93227784	0.01692180
5.81798996	0.04414047
5.85447805	0.03319706
5.94202521	−0.00099947
4.42152959	−0.00072826
3.32211694	0.02363822
4.49411603	0.00982052
4.80968966	0.00846967
3.81108707	−0.01312902

This converges in 6 iterations with tolerance 5.00×10^{-2} in the l_∞ norm and $\|\mathbf{r}^{(6)}\|_\infty =$ 0.046.

(c) All tolerances lead to the same convergence specifications.

12. With $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t \mathbf{y}$, we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t \mathbf{y} = \mathbf{y}^t \mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle; \qquad \text{(i)}$$

$$\langle c\mathbf{x}, \mathbf{y} \rangle = (c\mathbf{x})^t \mathbf{y} = c\mathbf{x}^t \mathbf{y} = c\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t c\mathbf{y} = \langle \mathbf{x}, c\mathbf{y} \rangle; \qquad \text{(ii)}$$

$$\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = (\mathbf{x} + \mathbf{z})^t \mathbf{y} = \left(\mathbf{x}^t + \mathbf{z}^t\right) \mathbf{y} = \mathbf{x}^t \mathbf{y} + \mathbf{z}^t \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle; \qquad \text{(iii)}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^t \mathbf{x} = \|\mathbf{x}\|_2^2 \geq 0. \qquad \text{(iv)}$$

We show (v) as follows:

If $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|_2^2 = 0$, then $\mathbf{x} = \mathbf{0}$ by the properties of norms. If $\mathbf{x} = \mathbf{0}$, then $0 = \mathbf{x}^t \mathbf{x} = \langle \mathbf{x}, \mathbf{x} \rangle$.

13. (a) Let $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ be a set of nonzero A-orthogonal vectors for the symmetric positive definite matrix A. Then $\langle \mathbf{v}^{(i)}, A\mathbf{v}^{(j)} \rangle = 0$, if $i \neq j$. Suppose

$$c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)} + \cdots + c_n\mathbf{v}^{(n)} = \mathbf{0},$$

where not all c_i are zero. Suppose k is the smallest integer for which $c_k \neq 0$. Then

$$c_k\mathbf{v}^{(k)} + c_{k+1}\mathbf{v}^{(k+1)} + \cdots + c_n\mathbf{v}^{(n)} = \mathbf{0}.$$

We solve for $\mathbf{v}^{(k)}$ to obtain

$$\mathbf{v}^{(k)} = -\frac{c_{k+1}}{c_k}\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}\mathbf{v}^{(n)}.$$

Multiplying by A gives

$$A\mathbf{v}^{(k)} = -\frac{c_{k+1}}{c_k}A\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}A\mathbf{v}^{(n)},$$

so

$$\begin{aligned}(\mathbf{v}^{(k)})^t A\mathbf{v}^{(k)} &= -\frac{c_{k+1}}{c_k}(\mathbf{v}^{(k)})^t A\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}(\mathbf{v}^{(k)t})A\mathbf{v}^{(n)} \\ &= -\frac{c_{k+1}}{c_k}\langle \mathbf{v}^{(k)}, A\mathbf{v}^{(k+1)} \rangle - \cdots - \frac{c_n}{c_k}\langle \mathbf{v}^{(k)}, A\mathbf{v}^{(n)} \rangle \\ &= -\frac{c_{k+1}}{c_k} \cdot 0 - \cdots - \frac{c_n}{c_k} \cdot 0 = 0.\end{aligned}$$

Since A is positive definite, $\mathbf{v}^{(k)} = \mathbf{0}$, which is a contradiction. Thus, all the c_i must be zero, and $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ is linearly independent.

(b) Let $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ be a set of nonzero A-orthogonal vectors for the symmetric positive definite matrix A, and let $\mathbf{z}$ be orthogonal to $\mathbf{v}^{(i)}$, for each $i = 1, \ldots, n$. From part (a), the set $\{\mathbf{v}^{(1)}, \ldots \mathbf{v}^{(n)}\}$ is linearly independent, so there is a collection of constants $\beta_1, \ldots, \beta_n$ with

$$\mathbf{z} = \sum_{i=1}^{n} \beta_i \mathbf{v}^{(i)}.$$

Hence,

$$\langle \mathbf{z}, \mathbf{z} \rangle = \mathbf{z}^t\mathbf{z} = \sum_{i=1}^{n} \beta_i \mathbf{z}^t\mathbf{v}^{(i)} = \sum_{i=1}^{n} \beta_i \cdot 0 = 0,$$

and Theorem 7.30, part (e), implies that $\mathbf{z} = \mathbf{0}$.

14. To prove Theorem 7.33 by mathematical induction:

(a) First note that we have

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + t_1\mathbf{v}^{(1)} = \mathbf{x}^{(0)} + \frac{\left\langle \mathbf{v}^{(1)}, \mathbf{r}^{(0)} \right\rangle}{\left\langle \mathbf{v}^{(1)}, A\mathbf{v}^{(1)} \right\rangle}\mathbf{v}^{(1)}.$$

Thus

$$A\mathbf{x}^{(1)} = A\mathbf{x}^{(0)} + \frac{\left\langle \mathbf{v}^{(1)}, \mathbf{r}^{(0)} \right\rangle}{\left\langle \mathbf{v}^{(1)}, A\mathbf{v}^{(1)} \right\rangle}A\mathbf{v}^{(1)}$$

and

$$\mathbf{b} - A\mathbf{x}^{(1)} = \mathbf{b} - A\mathbf{x}^{(0)} - \frac{\left\langle \mathbf{v}^{(1)}, \mathbf{r}^{(0)} \right\rangle}{\left\langle \mathbf{v}^{(1)}, A\mathbf{v}^{(1)} \right\rangle}A\mathbf{v}^{(1)}.$$

Hence,

$$\mathbf{r}^{(1)} = \mathbf{r}^{(0)} - \frac{\left\langle \mathbf{v}^{(1)}, \mathbf{r}^{(0)} \right\rangle}{\left\langle \mathbf{v}^{(1)}, A\mathbf{v}^{(1)} \right\rangle}A\mathbf{v}^{(1)}.$$

Taking the inner product with $\mathbf{v}^{(1)}$ gives

$$\begin{aligned}\left\langle \mathbf{r}^{(1)}, \mathbf{v}^{(1)} \right\rangle &= \left\langle \mathbf{r}^{(0)}, \mathbf{v}^{(1)} \right\rangle - \frac{\left\langle \mathbf{v}^{(1)}, \mathbf{r}^{(0)} \right\rangle}{\left\langle \mathbf{v}^{(1)}, A\mathbf{v}^{(1)} \right\rangle}\left\langle A\mathbf{v}^{(1)}, \mathbf{v}^{(1)} \right\rangle \\ &= \left\langle \mathbf{r}^{(0)}, \mathbf{v}^{(1)} \right\rangle - \left\langle \mathbf{v}^{(1)}, \mathbf{r}^{(0)} \right\rangle = 0.\end{aligned}$$

This establishes the base step.

(b) For the inductive hypothesis we assume that $\langle \mathbf{r}^{(k)}, \mathbf{v}^{(j)} \rangle = 0$, for all $k \le l$ and $j = 1, 2, \ldots, k$. We must then show

$$\left\langle \mathbf{r}^{(l+1)}, \mathbf{v}^{(j)} \right\rangle = 0, \quad \text{for} \quad j = 1, 2, \ldots, l+1.$$

We do this in two parts.
First, for $j = 1, 2, \ldots, l$, we will show that $\langle \mathbf{r}^{(l+1)}, \mathbf{v}^{(j)} \rangle = 0$. We have

$$\begin{aligned} \mathbf{x}^{(l+1)} &= \mathbf{x}^{(l)} + t_{l+1}\mathbf{v}^{(l+1)} \\ &= \mathbf{x}^{(l)} + \frac{\langle \mathbf{v}^{(l+1)}, \mathbf{r}^{(l)} \rangle}{\langle \mathbf{v}^{(l+1)}, A\mathbf{v}^{(l+1)} \rangle} \mathbf{v}^{(l+1)}, \end{aligned}$$

so

$$A\mathbf{x}^{(l+1)} = A\mathbf{x}^{(l)} + \frac{\langle \mathbf{v}^{(l+1)}, \mathbf{r}^{(l)} \rangle}{\langle \mathbf{v}^{(l+1)}, A\mathbf{v}^{(l+1)} \rangle} A\mathbf{v}^{(l+1)}.$$

Subtracting b from both sides gives

$$(2) \qquad -\mathbf{r}^{(l+1)} = -\mathbf{r}^{(l)} + \frac{\langle \mathbf{v}^{(l+1)}, \mathbf{r}^{(l)} \rangle}{\langle \mathbf{v}^{(l+1)}, A\mathbf{v}^{(l+1)} \rangle} A\mathbf{v}^{(l+1)}.$$

Taking the inner product of both sides of (2) with $\mathbf{v}^{(i)}$ gives

$$(3) \qquad -\left\langle \mathbf{r}^{(l+1)}, \mathbf{v}^{(i)} \right\rangle = -\left\langle \mathbf{r}^{(l)}, \mathbf{v}^{(i)} \right\rangle + \frac{\langle \mathbf{v}^{(l+1)}, \mathbf{r}^{(l)} \rangle}{\langle \mathbf{v}^{(l+1)}, A\mathbf{v}^{(l+1)} \rangle} \left\langle A\mathbf{v}^{(l+1)}, \mathbf{v}^{(i)} \right\rangle.$$

The first term on the right-hand side of (3) is 0 by the inductive hypothesis, and the factor $\langle A\mathbf{v}^{(l+1)}, \mathbf{v}^{(i)} \rangle$ is 0 because of A-orthogonality. Thus, $\langle \mathbf{r}^{(l+1)}, \mathbf{v}^{(i)} \rangle = 0$, for $1, 2, \ldots, l$.

(c) For the second part we take the inner product of both sides of (2) with $\mathbf{v}^{(l+1)}$ to obtain

$$-\left\langle \mathbf{r}^{(l+1)}, \mathbf{v}^{(l+1)} \right\rangle = -\left\langle \mathbf{r}^{(l)}, \mathbf{v}^{(l+1)} \right\rangle + \frac{\langle \mathbf{v}^{(l+1)}, \mathbf{r}^{(l)} \rangle}{\langle \mathbf{v}^{(l+1)}, A\mathbf{v}^{(l+1)} \rangle} \left\langle A\mathbf{v}^{(l+1)}, \mathbf{v}^{(l+1)} \right\rangle.$$

Thus,

$$-\left\langle \mathbf{r}^{(l+1)}, \mathbf{v}^{(l+1)} \right\rangle = -\left\langle \mathbf{r}^{(l)}, \mathbf{v}^{(l+1)} \right\rangle + \left\langle \mathbf{v}^{(l+1)}, \mathbf{r}^{(l)} \right\rangle = 0.$$

This completes the proof by induction.

15. If A is a positive definite matrix whose eigenvalues are $0 < \lambda_1 \le \cdots \le \lambda_n$, then $||A||_2 = \lambda_n$ and $||A^{-1}||_2 = 1/\lambda_1$, so $K_2(A) = \lambda_n/\lambda_1$.
For the matrix A in Example 3 we have

$$K_2(A) = \frac{\lambda_5}{\lambda_1} = \frac{700.031}{0.0570737} = 12265.2,$$

and the matrix AH has

$$K_2(AH) = \frac{\lambda_5}{\lambda_1} = \frac{1.88052}{0.156370} = 12.0261.$$

Maple gives $ConditionNumber(A, 2) = 12265.15914$ and $ConditionNumber(AH, 2) = 12.02598124$.