1. Two iterations of the SOR method with $\omega = 1.1$ give the following results.

   (a) $\mathbf{x}^{(2)} = (1.512775000, -0.8298491667, -0.0843373667)^t$

   (b) $\mathbf{x}^{(2)} = (-1.58523750, 1.37885688, -0.7039212812)^t$

   (c) $\mathbf{x}^{(2)} = (-0.6604902, 0.03700749, -0.2493513, 0.6561139)^t$

   (d) $\mathbf{x}^{(2)} = (0.3781250000, 1.445468750, 0.3596914062, 1.458531250, 0.3071921875, 1.572124727)^t$

2. Two iterations of the SOR method with $\omega = 1.1$ give the following results.

   (a) $\mathbf{x}^{(2)} = (0.05410079, -0.2115435, 0.6477159)^t$

   (b) $\mathbf{x}^{(2)} = (0.9876790, 0.9784935, 0.7899328)^t$

   (c) $\mathbf{x}^{(2)} = (-0.71885, 2.818822, -0.2809726, -2.235422)^t$

   (d) $\mathbf{x}^{(2)} = (1.079675, -1.260654, 2.042489, 1.995373, 2.049536)^t$

3. Two iterations of the SOR method with $\omega = 1.3$ give the following results.

   (a) $\mathbf{x}^{(2)} = (1.455783334, -0.7721494442, -0.0805396228)^t$

   (b) $\mathbf{x}^{(2)} = (-1.42073750, 1.595758125, -0.8597927812)^t$

   (c) $\mathbf{x}^{(2)} = (-0.7268893, 0.1251483, -0.2923371, 0.7037018)^t$

   (d) $\mathbf{x}^{(2)} = (0.5281250000, 1.480781250, 0.322816406, 1.359718750, 0.4288171875, 1.505949961)^t$

4. Two iterations of the SOR method with $\omega = 1.3$ give the following results.

   (a) $\mathbf{x}^{(2)} = (-0.1040103, -0.1331814, 0.6774997)^t$

   (b) $\mathbf{x}^{(2)} = (0.957073, 0.9903875, 0.7206569)^t$

   (c) $\mathbf{x}^{(2)} = (-1.23695, 3.228752, -0.1523888, -2.041266)^t$

   (d) $\mathbf{x}^{(2)} = (0.7064258, -0.4103876, 2.417063, 2.251955, 1.061507)^t$

5. The SOR Algorithm with $\omega = 1.2$ gives the following results.

   (a) $\mathbf{x}^{(8)} = (1.447503814, -0.8359297624, -0.0445516532)^t$

   (b) $\mathbf{x}^{(6)} = (-1.454582850, 1.454498863, -0.7273302714)^t$

   (c) $\mathbf{x}^{(6)} = (-0.75308971, 0.04117281, -0.28074817, 0.69163506)^t$

   (d) $\mathbf{x}^{(7)} = (0.3571284945, 1.428582240, 0.3571489731, 1.571440116, 0.2857000650, 1.571445036)^t$

6. The SOR Algorithm with $\omega = 1.2$ gives the following results.

   (a) $\mathbf{x}^{(11)} = (0.03544356, -0.23718333, 0.65788317)^t$

   (b) $\mathbf{x}^{(7)} = (0.9958341, 0.9579041, 0.7915756)^t$

   (c) $\mathbf{x}^{(8)} = (-0.7976009, 2.795288, -0.2588293, -2.251768)^t$

   (d) $\mathbf{x}^{(10)} = (0.7866310, -1.002807, 1.866530, 1.912645, 1.989792)^t$

7. The tridiagonal matrix is in part (d).
(10d): For $\omega = 1.033370453$ we have

$$\mathbf{x}^{(5)} = (0.3571407017, 1.428570817, 0.357142771, 1.571421010, 0.2857118407, 1.571428256)^t.$$

8. The tridiagonal matrices are in parts (b) and (c).
(9b): For $\omega = 1.012823$ we have $\mathbf{x}^{(4)} = (0.9957846, 0.9578935, 0.7915788)^t$.
(9c): For $\omega = 1.153499$ we have $\mathbf{x}^{(7)} = (-0.7977651, 2.795343, -0.2588021, -2.251760)^t$.

9. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of T_ω. Then

$$\begin{aligned}\prod_{i=1}^{n} \lambda_i &= \det T_\omega = \det\left((D-\omega L)^{-1}[(1-\omega)D + \omega U]\right)\\ &= \det(D-\omega L)^{-1} \det((1-\omega)D+\omega U) = \det\left(D^{-1}\right)\det((1-\omega)D)\\ &= \left(\frac{1}{(a_{11}a_{22}\ldots a_{nn})}\right)\left((1-\omega)^n a_{11}a_{22}\ldots a_{nn}\right) = (1-\omega)^n.\end{aligned}$$

Thus

$$\rho(T_\omega) = \max_{1\le i\le n} |\lambda_i| \ge |\omega - 1|,$$

and $|\omega - 1| < 1$ if and only if $0 < \omega < 2$.

10. (a) The system was reordered so that the diagonal of the matrix had nonzero entries.
(b) (i) The solution vector using 30 iterations is

$$(0.00362, -6339.744638, -3660.253272, -8965.755808, 6339.744638, 10000, -7320.508959, 6339.746729)^t.$$

(ii) The solution vector using 57 iterations is

$$(-0.002651, -6339.744637, -3660.255362, -8965.752851, 6339.748259, 10000, -7320.506544, 6339.748258)^t.$$

(iii) Method does not converge using $\omega = 1.25$. However, using $\omega = 1.1$ and using 132 iterations gives the solution vector

$$(0.0045175, -6339.744528, -3660.253009, -8965.756179, 6339.743756, 10000, -7320.509547, 6339.747544)^t.$$

11. The results that follow include approximations from the Jacobi and Gauss-Seidel methods for comparison.

	Jacobi 33 iterations	Gauss-Seidel 8 iterations	SOR ($\omega = 1.2$) 13 iterations		Jacobi iterations	Gauss-Seidel iterations	SOR ($\omega = 1.2$) iterations
x_1	1.53873501	1.53873270	1.53873549	x_{41}	0.02185033	0.02184781	0.02184855
x_2	0.73142167	0.73141966	0.73142226	x_{42}	0.02133203	0.02132965	0.02133038
x_3	0.10797136	0.10796931	0.10797063	x_{43}	0.02083782	0.02083545	0.02083615
x_4	0.17328530	0.17328340	0.17328480	x_{44}	0.02036585	0.02036360	0.02036429
x_5	0.04055865	0.04055595	0.04055737	x_{45}	0.01991483	0.01991261	0.01991324
x_6	0.08525019	0.08524787	0.08524925	x_{46}	0.01948325	0.01948113	0.01948175
x_7	0.16645040	0.16644711	0.16644868	x_{47}	0.01907002	0.01906793	0.01906846
x_8	0.12198156	0.12197878	0.12198026	x_{48}	0.01867387	0.01867187	0.01867239
x_9	0.10125265	0.10124911	0.10125043	x_{49}	0.01829386	0.01829190	0.01829233
x_{10}	0.09045966	0.09045662	0.09045793	x_{50}	0.71792896	0.01792707	0.01792749
x_{11}	0.07203172	0.07202785	0.07202912	x_{51}	0.01757833	0.01757648	0.01757683
x_{12}	0.07026597	0.07026266	0.07026392	x_{52}	0.01724113	0.01723933	0.01723968
x_{13}	0.06875835	0.06875421	0.06875546	x_{53}	0.01691660	0.01691487	0.01691517
x_{14}	0.06324659	0.06324307	0.06324429	x_{54}	0.01660406	0.01660237	0.01660267
x_{15}	0.05971510	0.05971083	0.05971200	x_{55}	0.01630279	0.01630127	0.01630146
x_{16}	0.05571199	0.05570834	0.05570949	x_{56}	0.01601230	0.01601082	0.01601101
x_{17}	0.05187851	0.05187416	0.05187529	x_{57}	0.01573198	0.01573087	0.01573077
x_{18}	0.04924911	0.04924537	0.04924648	x_{58}	0.01546129	0.01546020	0.01546010
x_{19}	0.04678213	0.04677776	0.04677885	x_{59}	0.01519990	0.01519909	0.01519878
x_{20}	0.04448679	0.04448303	0.04448409	x_{60}	0.01494704	0.01494626	0.01494595
x_{21}	0.04246924	0.04246493	0.04246597	x_{61}	0.01470181	0.01470085	0.01470077
x_{22}	0.04053818	0.04053444	0.04053546	x_{62}	0.01446510	0.01446417	0.01446409
x_{23}	0.03877273	0.03876852	0.03876952	x_{63}	0.01423556	0.01423437	0.01423461
x_{24}	0.03718190	0.03717822	0.03717920	x_{64}	0.01401350	0.01401233	0.01401256
x_{25}	0.03570858	0.03570451	0.03570548	x_{65}	0.01380328	0.01380234	0.01380242
x_{26}	0.03435107	0.03434748	0.03434844	x_{66}	0.01359448	0.01359356	0.01359363
x_{27}	0.03309542	0.03309152	0.03309246	x_{67}	0.01338495	0.01338434	0.01338418
x_{28}	0.03192212	0.03191866	0.03191958	x_{68}	0.01318840	0.01318780	0.01318765
x_{29}	0.03083007	0.03082637	0.03082727	x_{69}	0.01297174	0.01297109	0.01297107
x_{30}	0.02980997	0.02980666	0.02980755	x_{70}	0.01278663	0.01278598	0.01278597
x_{31}	0.02885510	0.02885160	0.02885248	x_{71}	0.01270328	0.01270263	0.01270271
x_{32}	0.02795937	0.02795621	0.02795707	x_{72}	0.01252719	0.01252656	0.01252663
x_{33}	0.02711787	0.02711458	0.02711543	x_{73}	0.01237700	0.01237656	0.01237654
x_{34}	0.02632478	0.02632179	0.02632262	x_{74}	0.01221009	0.01220965	0.01220963
x_{35}	0.02557705	0.02557397	0.02557479	x_{75}	0.01129043	0.01129009	0.01129008
x_{36}	0.02487017	0.02486733	0.02486814	x_{76}	0.01114138	0.01114104	0.01114102
x_{37}	0.02420147	0.02419858	0.02419938	x_{77}	0.01217337	0.01217312	0.01217313
x_{38}	0.02356750	0.02356482	0.02356560	x_{78}	0.01201771	0.01201746	0.01201746
x_{39}	0.02296603	0.02296333	0.02296410	x_{79}	0.01542910	0.01542896	0.01542896
x_{40}	0.02239424	0.02239171	0.02239247	x_{80}	0.01523810	0.01523796	0.01523796

12. For $0 < \omega < 2$, let $T_\omega = (D - \omega L)^{-1}\left[(1-\omega)D + \omega L^t\right]$. Let $P = A - T_\omega^t A T_\omega$ and note that P is symmetric.

As in Exercise 17 of Section 7.3, we derive a new representation for T_ω:

$$(D - \omega L)T_\omega = (1-\omega)D + \omega L^t = (D - \omega L) - \omega A, \quad \text{so} \quad T_\omega = I - \omega(D - \omega L)^{-1}A.$$

Let

$$Q = \omega(D - \omega L)^{-1}A \quad \text{and} \quad Q^t = \omega A\left[(D - \omega L)^{-1}\right]^t.$$

We again have

$$P = Q^t\left[AQ^{-1} + \left(Q^t\right)^{-1}A - A\right]Q.$$

But

$$AQ^{-1} = \frac{1}{\omega}(D - \omega L) \quad \text{and} \quad \left(Q^t\right)^{-1}A = \frac{1}{\omega}\left(D - \omega L^t\right)$$

so

$$\begin{aligned} AQ^{-1} + \left(Q^t\right)^{-1}A - A &= \frac{1}{\omega}\left[D - \omega L + D - \omega L^t\right] - A \\ &= \frac{2}{\omega}D - D + D - L - L^t - A \\ &= \left(\frac{2}{\omega} - 1\right)D. \end{aligned}$$

Thus $P = \left(\frac{2}{\omega} - 1\right)Q^t D Q$. Since $0 < \omega < 2$, we have $\frac{2}{\omega} - 1 > 0$ and P is positive definite.

The proof follows Exercise 17 with T_g replaced by T_ω. Hence, T_ω is convergent.