1. (a) The eigenvalue $\lambda_1 = 0$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = -1$ has the eigenvector $\mathbf{x}_2 = (1, -2)^t$.

   (b) The eigenvalue $\lambda_1 = -1/3$ has the eigenvector $\mathbf{x}_1 = (-3/2, 1)^t$, and the eigenvalue $\lambda_2 = -1/2$ has the eigenvector $\mathbf{x}_2 = (-2, 1)^t$.

   (c) The eigenvalue $\lambda_1 = -1$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = 4$ has the eigenvector $\mathbf{x}_2 = (4, 1)^t$.

   (d) The eigenvalue $\lambda_1 = 3$ has the eigenvector $\mathbf{x}_1 = (-1, 1, 2)^t$, the eigenvalue $\lambda_2 = 4$ has the eigenvector $\mathbf{x}_2 = (0, 1, 2)^t$, and the eigenvalue $\lambda_3 = -2$ has the eigenvector $\mathbf{x} = (-3, 8, 1)^t$.

   (e) The eigenvalue $\lambda_1 = \lambda_2 = 1/2$ has the eigenvector $\mathbf{x}_1 = (0, 5, 12)^t$, and the eigenvalue $\lambda_3 = -1/3$ has the eigenvector $\mathbf{x}_3 = (0, 0, 1)^t$.

   (f) The eigenvalue $\lambda_1 = 2 + 2i$ has the eigenvector $\mathbf{x}_1 = (0, -2i, 1)^t$, the eigenvalue $\lambda_2 = 2 - 2i$ has the eigenvector $\mathbf{x}_2 = (0, 2i, 1)^t$, and the eigenvalue $\lambda_3 = 2$ has the eigenvector $\mathbf{x}_3 = (1, 0, 0)^t$.

2. (a) The eigenvalue $\lambda_1 = 3$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = 1$ has the eigenvector $\mathbf{x}_2 = (1, 1)^t$.

   (b) The eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$ has the eigenvector $\mathbf{x} = \left(1, (1+\sqrt{5})/2\right)^t$, and the eigenvalue $\lambda_2 = \frac{1-\sqrt{5}}{2}$ has the eigenvector $\mathbf{x} = \left(1, (1-\sqrt{5})/2\right)^t$.

   (c) The eigenvalue $\lambda_1 = \frac{1}{2}$ has the eigenvector $\mathbf{x}_1 = (1, 1)^t$, and the eigenvalue $\lambda_2 = -\frac{1}{2}$ has the eigenvector $\mathbf{x}_2 = (1, -1)^t$.

   (d) The eigenvalue $\lambda_1 = 1$ has the eigenvector $\mathbf{x}_1 = (1, -1, 0)^t$, and the eigenvalue $\lambda_2 = \lambda_3 = 3$ has the eigenvectors $\mathbf{x}_2 = (1, 1, 0)^t$ and $\mathbf{x}_3 = (0, 0, 1)^t$.

   (e) The eigenvalue $\lambda_1 = 7$ has the eigenvector $\mathbf{x}_1 = (1, 4, 4)^t$, the eigenvalue $\lambda_2 = -1$ has the eigenvector $\mathbf{x}_2 = (1, 0, 0)^t$, and the eigenvalue $\lambda_3 = 3$ has the eigenvector $\mathbf{x}_3 = (1, 2, 0)^t$.

   (f) The eigenvalue $\lambda_1 = \lambda_2 = 1$ has the eigenvectors $\mathbf{x}_1 = (-1, 0, 1)^t$ and $\mathbf{x}_2 = (-1, 1, 0)^t$, and the eigenvalue $\lambda_3 = 5$ has the eigenvector $\mathbf{x} = (1, 2, 1)^t$.

3. (a) The eigenvalue $\lambda_1 = 2 + \sqrt{2}i$ has the eigenvector $(\sqrt{2}i, -1)^t$ and the eigenvalue $\lambda_2 = 2 - \sqrt{2}i$ has the eigenvector $(\sqrt{2}i, 1)^t$.

   (b) The eigenvalue $\lambda_1 = \frac{1}{2}(3 + \sqrt{7}i)$ has the eigenvector $(1 - \sqrt{7}i, 2)^t$ and the eigenvalue $\lambda_2 = \frac{1}{2}(3 - \sqrt{7}i$ has the eigenvector $(1 + \sqrt{7}i, 2)^t$.

4. (a) The eigenvalue $\lambda_1 = 1 + \sqrt{3}i$ has the eigenvector $(2\sqrt{3}i, -\sqrt{3}i, -3)^t$ and the eigenvalue $\lambda_2 = 1 - \sqrt{3}i$ has the eigenvector $(2\sqrt{3}i, -\sqrt{3}i, 3)^t$.

   (b) The eigenvalue $\lambda_1 = 1+\sqrt{2}i$ has the eigenvector $(3, 1-\sqrt{2}i, -1-2\sqrt{2}i)^t$ and the eigenvalue $\lambda_2 = 1 - \sqrt{2}i$ has the eigenvector $(3, 1 + \sqrt{2}i, -1 + 2\sqrt{2}i)^t$.

5. The spectral radii for the matrices in Exercise 2 are:

   (a) 1 (b) $\frac{1}{2}$ (c) 4 (d) 4 (e) $\frac{1}{2}$ (f) $2\sqrt{2}$

6. The spectral radii for the matrices in Exercise 1 are;

(a) 3 (b) $\frac{1+\sqrt{5}}{2}$ (c) $\frac{1}{2}$ (d) 3 (e) 7 (f) 5

7. Only the matrices in 2(b) and 2(e) are convergent.

8. Only the matrix in 1(c) is convergent.

9. The l_2 norms for the matrices in Exercise 1 are:

(a) 3.162278 (b) 1.458020 (c) 5.036796 (d) 5.601152 (e) 2.896954 (f) 4.701562

10. The l_2 norms for the matrices in Exercise 1 are:

(a) 3 (b) 1.618034 (c) 0.5 (d) 3 (e) 8.224257 (f) 5.203527

11. Since

$$A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{2^k - 1}{2^{k+1}} & 2^{-k} \end{bmatrix}, \quad \text{we have} \quad \lim_{k\to\infty} A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Also,

$$A_2^k = \begin{bmatrix} 2^{-k} & 0 \\ \frac{16k}{2^{k-1}} & 2^{-k} \end{bmatrix}, \quad \text{so} \quad \lim_{k\to\infty} A_2^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

12. Suppose the λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Then $A\mathbf{x} = \lambda\mathbf{x}$, $A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$, and in general, $A^m\mathbf{x} = \lambda^m\mathbf{x}$, for any positive integer m. As a consequence,

$$0 = A^m\mathbf{x} = \lambda^m\mathbf{x} \quad \text{which implies that } \lambda^m = 0.$$

Hence $\lambda = 0$.

13. Let A be an $n \times n$ matrix. Expanding across the first row gives the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)M_{11} + \sum_{j=2}^{n}(-1)^{j+1} a_{1j} M_{1j}.$$

The determinants M_{1j} are of the form

$$M_{1j} = \det \begin{bmatrix} a_{21} & a_{22} - \lambda & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,j-1} - \lambda & a_{j-1,j+1} & \cdots & a_{j-1,n} \\ a_{j,1} & a_{j,2} & \cdots & a_{j,j-1} & a_{j,j+1} & \cdots & a_{j,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,j-1} & a_{j+1,j+1} - \lambda & \cdots & a_{j+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} - \lambda \end{bmatrix},$$

for $j = 2, \ldots, n$. Note that each M_{1j} has $n-2$ entries of the form $a_{ii} - \lambda$. Thus,

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)M_{11} + \{\text{terms of degree } n-2 \text{ or less}\}.$$

Since

$$M_{11} = \det \begin{bmatrix} a_{22} - \lambda & a_{23} & \cdots & \cdots & a_{2n} \\ a_{32} & a_{33} - \lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1,n} \\ a_{n2} & \cdots & \cdots & a_{n,n-1} & a_{nn} - \lambda \end{bmatrix}$$

is of the same form as $\det(A - \lambda I)$, the same argument can be repeatedly applied to determine

$$p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) + \{\text{terms of degree } n-2 \text{ or less in } \lambda\}.$$

Thus, $p(\lambda)$ is a polynomial of degree n.

14. (a) $P(\lambda) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda) = \det(A - \lambda I)$, so $P(0) = \lambda_1 \cdots \lambda_n = \det A$.

(b) A singular if and only if $\det A = 0$, which is equivalent to at least one of λ_i being 0.

15. (a) $\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I)$

(b) If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$, and by induction, $A^k\mathbf{x} = \lambda^k\mathbf{x}$.

(c) If $A\mathbf{x} = \lambda\mathbf{x}$ and A^{-1} exists, then $\mathbf{x} = \lambda A^{-1}\mathbf{x}$. By Exercise 14 (b), $\lambda \neq 0$, so $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$.

(d) Since $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$, we have $(A^{-1})^2\mathbf{x} = \frac{1}{\lambda}A^{-1}\mathbf{x} = \frac{1}{\lambda^2}\mathbf{x}$. Mathematical induction gives

$$(A^{-1})^k\mathbf{x} = \frac{1}{\lambda^k}\mathbf{x}.$$

(e) If $A\mathbf{x} = \lambda\mathbf{x}$, then

$$q(A)\mathbf{x} = q_0\mathbf{x} + q_1A\mathbf{x} + \ldots + q_kA^k\mathbf{x} = q_0\mathbf{x} + q_1\lambda\mathbf{x} + \ldots + q_k\lambda^k\mathbf{x} = q(\lambda)\mathbf{x}.$$

(f) Let $A - \alpha I$ be nonsingular. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$(A - \alpha I)\mathbf{x} = A\mathbf{x} - \alpha I\mathbf{x} = \lambda\mathbf{x} - \alpha\mathbf{x} = (\lambda - \alpha)\mathbf{x}.$$

Thus,

$$\frac{1}{\lambda - \alpha}\mathbf{x} = (A - \alpha I)^{-1}\mathbf{x}.$$

16. Since $A^tA = A^2$ and $A\mathbf{x} = \lambda\mathbf{x}$, we have $A^2\mathbf{x} = \lambda^2\mathbf{x}$. Thus

$$\rho\left(A^tA\right) = \rho\left(A^2\right) = [\rho(A)]^2 \quad \text{and} \quad \|A\|_2 = \left[\rho\left(A^tA\right)\right]^{\frac{1}{2}} = \rho(A).$$

17. (a) We have the real eigenvalue $\lambda = 1$ with the eigenvector $\mathbf{x} = (6, 3, 1)^t$.

(b) Choose any multiple of the vector $(6, 3, 1)^t$.

18. For

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

we have $\rho(A) = \rho(B) = 1$ and $\rho(A + B) = 3$.

19. Let $A\mathbf{x} = \lambda\mathbf{x}$. Then $|\lambda|\ \|\mathbf{x}\| = \|A\mathbf{x}\| \leq \|A\|\ \|\mathbf{x}\|$, which implies $|\lambda| \leq \|A\|$. Also, $(1/\lambda)\mathbf{x} = A^{-1}\mathbf{x}$ so $1/|\lambda| \leq \|A^{-1}\|$ and $\|A^{-1}\|^{-1} \leq |\lambda|$.