1. (i) Matrices (a) and (c) are symmetric.
   (ii) Matrices (a), (b), and (c) are nonsingular.
   (iii) Matrices (a) and (b) are strictly diagonally dominant.
   (iv) Matrices (b) and (c) are positive definite.

2. (i) The only symmetric matrix is (a).
   (ii) All are nonsingular.
   (iii) Matrices (a) and (b) are strictly diagonally dominant.
   (iv) The only positive definite matrix is (a).

3. The LDL^t factorization of the matrices A have the following forms.

(a) $L = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.09090909 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2.75 & 0 \\ 0 & 0 & 1.72727273 \end{bmatrix}$

(b) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0.2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3.6 \end{bmatrix}$

(c) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & -0.3333333 & 1 & 0 \\ 0.25 & 0.3333333 & 0.6.0714286 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4.666667 & 0 \\ 0 & 0 & 0 & 5.696429 \end{bmatrix}$

(d) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0.25 & -0.9090909 & 1 & 0 \\ 0.25 & -0.4545455 & 0.3684211 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2.75 & 0 & 0 \\ 0 & 0 & 1.727273 & 0 \\ 0 & 0 & 0 & 2.947368 \end{bmatrix}$

4. The LDL^t factorization of the matrices A have the following forms.

(a) $L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$

(b) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0.25 & -0.45454545 & 1 & 0 \\ 0.25 & 0.27272727 & 0.076923077 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4.0 & 0 & 0 & 0 \\ 0 & 2.75 & 0 & 0 \\ 0 & 0 & 1.1818182 & 0 \\ 0 & 0 & 0 & 1.5384615 \end{bmatrix}$

(c) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ -0.25 & -0.27272727 & 1 & 0 \\ 0 & 0 & 0.44 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2.75 & 0 & 0 \\ 0 & 0 & 4.5454545 & 0 \\ 0 & 0 & 0 & 3.12 \end{bmatrix}$

(d) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.33333333 & 1 & 0 & 0 \\ 0.16666667 & 0.2 & 1 & 0 \\ -0.16666667 & 0.1 & -0.24324324 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 3.3333333 & 0 & 0 \\ 0 & 0 & 3.7 & 0 \\ 0 & 0 & 0 & 2.5810811 \end{bmatrix}$

5. Cholesky's Algorithm gives the following results.

(a) $L = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{11}/2 & 0 \\ 1/2 & \sqrt{11}/22 & \sqrt{209}/11 \end{bmatrix}$

(b) $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 1 & \sqrt{5}/5 & \sqrt{95}/5 \end{bmatrix}$

(c) $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ 1 & -\sqrt{3}/3 & \sqrt{42}/3 & 0 \\ 1/20 & \sqrt{3}/3 & 17\sqrt{42}/84 & \sqrt{4466}/28 \end{bmatrix}$

(d) $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1/2 & \sqrt{11}/2 & 0 & 0 \\ 1/2 & -\sqrt{11}/22 & \sqrt{209}/11 & 0 \\ 1/2 & -5\sqrt{11}/22 & 7\sqrt{209}/209 & 2\sqrt{266}/19 \end{bmatrix}$

6. Cholesky's Algorithm gives the following results.

(a) $L = \begin{bmatrix} 1.414213 & 0 & 0 \\ -0.7071069 & 1.224743 & 0 \\ 0 & -0.8164972 & 1.154699 \end{bmatrix}$

(b) $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0.5 & 1.658311 & 0 & 0 \\ 0.5 & -0.7537785 & 1.087113 & 0 \\ 0.5 & 0.4522671 & 0.08362442 & 1.240346 \end{bmatrix}$

(c) $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0.5 & 1.658311 & 0 & 0 \\ -0.5 & -0.4522671 & 2.132006 & 0 \\ 0 & 0 & 0.9380833 & 1.766351 \end{bmatrix}$

(d) $L = \begin{bmatrix} 2.449489 & 0 & 0 & 0 \\ 0.8164966 & 1.825741 & 0 & 0 \\ 0.4082483 & 0.3651483 & 1.923538 & 0 \\ -0.4082483 & 0.1825741 & -0.4678876 & 1.606574 \end{bmatrix}$

7. The modified factorization algorithm gives the following results.

(a) $x_1 = 1,\ x_2 = -1,\ x_3 = 0$

(b) $x_1 = 0.2,\ x_2 = -0.2,\ x_3 = -0.2,\ x_4 = 0.25$

(c) $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = 2$

(d) $x_1 = -0.8586387,\ x_2 = 2.418848,\ x_3 = -0.9581152,\ x_4 = -1.272251$

8. The modified factorization algorithm gives the following results.

(a) $x_1 = -13/19,\ x_2 = 21/19,\ x_3 = 54/19$

(b) $x_1 = -3/38,\ x_2 = 4/19,\ x_3 = -1/19$

(c) $x_1 = -452/319,\ x_2 = 373/319,\ x_3 = 763/319,\ x_4 = -356/319$

(d) $x_1 = 5/28,\ x_2 = 5/7,\ x_3 = 1/4,\ x_4 = 9/28$

9. The modified Cholesky's algorithm gives the following results.

(a) $x_1 = 1, x_2 = -1, x_3 = 0$

(b) $x_1 = 0.2, x_2 = -0.2, x_3 = -0.2, x_4 = 0.25$

(c) $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = 2$

(d) $x_1 = -0.85863874, x_2 = 2.4188482, x_3 = -0.95811518, x_4 = -1.2722513$

10. (a) $x_1 = -0.6842105265, x_2 = 1.105263158, x_3 = 2.842105263$
    (b) $x_1 = -0.07894736890, x_2 = 0.2105263158, x_3 = -0.05263157895$
    (c) $x_1 = -1.416927900, x_2 = 1.169278997, x_3 = 2.391849530, x_4 = -1.115987461$
    (d) $x_1 = 0.1785714286, x_2 = 0.7142857142, x_3 = 0.25, x_4 = 0.3214285714$

11. The Crout Factorization Algorithm gives the following results.
    (a) $x_1 = 0.5$, $x_2 = 0.5$, $x_3 = 1$
    (b) $x_1 = -0.9999995$, $x_2 = 1.999999$, $x_3 = 1$
    (c) $x_1 = 1$, $x_2 = -1$, $x_3 = 0$
    (d) $x_1 = -0.09357798$, $x_2 = 1.587156$, $x_3 = -1.167431$, $x_4 = 0.5412844$

12. The Crout Factorization Algorithm gives the following results.
    (a) $x_1 = 3.600000000$, $x_2 = -4.200000000$, $x_3 = 2.800000000$
    (b) $x_1 = 3.944444444$, $x_2 = 2.888888889$, $x_3 = -0.7222222222$
    (c) $x_1 = 2.380952381$, $x_2 = 1.761904762$, $x_3 = 1.904761905$, $x_4 = 2.047619048$
    (d) $x_1 = 0.6666666667$, $x_2 = 0.3333333334$, $x_3 = -0.6666666666$, $x_4 = -1.000000000$, $x_5 = 0.000000000$

13. We have $x_i = 1$, for each $i = 1, \ldots, 10$.

14. The modified LDL^t factorization gives the following results.

(a)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 1 & 4 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

(d)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 1 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

15. Only the matrix in (d) is positive definite.

16. When $\alpha > \frac{8}{7}$ the matrix is positive definite.

17. When $-2 < \alpha < \frac{3}{2}$ the matrix is positive definite.

18. When $0 < \beta < \frac{1}{2}$ and $\beta + 2 < |\alpha| < 3$ the matrix is strictly diagonally dominant.

19. When $0 < \beta < 1$ and $3 < \alpha < 5 - \beta$ the matrix is strictly diagonally dominant.

20. (a) Yes.

(b) Not necessarily. Consider $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

(c) Not necessarily. Consider $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

(d) Not necessarily. Consider $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

(e) Not necessarily. Consider $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

21. (a) No; for example, consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Yes, since $A = A^t$.

(c) Yes, since $\mathbf{x}^t(A + B)\mathbf{x} = \mathbf{x}^t A\mathbf{x} + \mathbf{x}^t B\mathbf{x}$.

(d) Yes, since $\mathbf{x}^t A^2 \mathbf{x} = \mathbf{x}^t A^t A\mathbf{x} = (A\mathbf{x})^t(A\mathbf{x}) \geq 0$, and because A is nonsingular, equality holds only if $\mathbf{x} = \mathbf{0}$.

(e) No; for example, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$.

22. (a) When $\alpha = 2$ the matrix is singular.

(b) The matrix A cannot be strictly diagonally dominant regardless of α.

(c) The matrix is symmetric for all values of α.

(d) The matrix is positive definite when $\alpha > 2$.

23. (a) Since $\det A = 3\alpha - 2\beta$, the matrix A is singular if and only if $\alpha = 2\beta/3$.

(b) The matrix is strictly diagonally dominant when $|\alpha| > 1$ and $|\beta| < 1$.

(c) The matrix is symmetric when $\beta = 1$.

(d) The matrix is positive definite when $\alpha > \frac{2}{3}$ and $\beta = 1$.

24. Yes, since $A^t B^t = (BA)^t = (AB)^t = B^t A^t$.

25. One example is $A = \begin{bmatrix} 1.0 & 0.2 \\ 0.1 & 1.0 \end{bmatrix}$.

26. Partition $A^{(k)}$ into the form

$$A^{(k)} = \left[\begin{array}{cccc|ccc} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1,k}^{(1)} & a_{1,k+1}^{(1)} & \dots & a_{l,n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2k}^{(2)} & a_{2,k+1}^{(2)} & \dots & a_{2,n}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{k,k}^{(k)} & a_{k,k+1}^{(k)} & \dots & a_{k,n}^{(k)} \\ \hline 0 & \dots & 0 & a_{k+1,k}^{(k)} & a_{k+1,k+1}^{(k)} & \dots & a_{k+1,n}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{n,k}^{(k)} & a_{n,k+1}^{(k)} & \dots & a_{n,n}^{(k)} \end{array}\right] = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}.$$

The multiplier matrix $M^{(k-1)}$ and $A^{(k-1)}$ can be similarly partitioned into

$$M^{(k-1)} = \left[\begin{array}{cccc|ccccc} 1 & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & \vdots & & & \vdots \\ 0 & \dots & 0 & -m_{k,k-1} & 1 & 0 & \dots & \dots & 0 \\ \hline 0 & \dots & 0 & -m_{k+1,k-1} & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -m_{n,k-1} & 0 & 0 & \dots & 0 & 1 \end{array}\right] = \begin{bmatrix} M_{11}^{(k-1)} & O \\ M_{21}^{(k-1)} & I \end{bmatrix},$$

where $M_{11}^{(k-1)}$ is a $k \times k$ lower triangular matrix, O is a $k \times (n-k)$ block of zeros, $M_{21}^{(k-1)}$ is an $(n-k) \times k$ matrix, I is an $(n-k) \times (n-k)$ identity matrix, and

$$A^{(k-1)} = \begin{bmatrix} A_{11}^{(k-1)} & A_{12}^{(k-1)} \\ A_{21}^{(k-1)} & A_{22}^{(k-1)} \end{bmatrix}.$$

Here $A_{11}^{(k-1)}$ is $k\times k$, $A_{12}^{(k-1)}$ is $k\times(n-k)$, $A_{21}^{(k-1)}$ is $(n-k)\times k$, and $A_{22}^{(k-1)}$ is $(n-k)\times(n-k)$. The formation of $A_{11}^{(k)}$ can be obtained from the partitioned product of $M^{(k-1)}$ and $A^{(k-1)}$ and is given by

$$A_{11}^{(k)} = M_{11}^{(k-1)}A_{11}^{(k-1)} + 0\cdot A_{21}^{(k-1)} = M_{11}^{(k-1)}A_{11}^{(k-1)}.$$

In a similar manner, each of $M^{(k-2)},\ldots,M^{(1)}$ and $A^{(k-2)},\ldots,A^{(1)}$ can be partitioned to obtain

$$A_{11}^{(k)} = M_{11}^{(k-1)}A_{11}^{(k-1)} = M_{11}^{(k-1)}M_{11}^{(k-2)}A_{11}^{(k-2)} = \ldots = M_{11}^{(k-1)}M_{11}^{(k-2)}\ldots M_{11}^{(1)}A_{11}^{(1)},$$

where $A_{11}^{(1)} = A_{11}$ is the $k\times k$ leading principal submatrix of A. Assume all leading principal submatrices of A are nonsingular. Then $a_{11}\neq 0$, and the elimination process can be started. For the inductive hypothesis, assume that $k-1$ elimination steps can be performed without row interchanges. It follows that $a_{11}^{(1)},\ldots,a_{k-1,k-1}^{(k-1)}$ are all nonzero and the above equation holds. Taking determinants produces

$$a_{11}^{(1)}a_{22}^{(2)}\ldots a_{k-1,k-1}^{(k-1)}a_{k,k}^{(k)} = \det A_{11}^{(k)} = \det M_{11}^{(k-1)}\det M_{11}^{(k-2)}\ldots\det M_{11}^{(1)}\det A_{11}\neq 0.$$

Hence, $a_{k,k}^{(k)}\neq 0$ and the process can continue. By mathematical induction all pivot elements $a_{11}^{(1)},\ldots,a_{n,n}^{(n)}$ are nonzero and Gaussian elimination can be performed without row interchanges.

Conversely, suppose Gaussian elimination can be performed without row interchanges. It follows that all the pivot elements $a_{11}^{(1)},\ldots,a_{n,n}^{(n)}$ are nonzero. Thus,

$$\det A_{11} = a_{11}^{(1)}a_{22}^{(2)}\ldots a_{k,k}^{(k)}\neq 0,$$

and the $k\times k$ principal leading submatrix is nonsingular, for each $k=1,2,\ldots,n$.

27. The Crout Factorization Algorithm can be rewritten as follows:

STEP 1 Set $l_1 = a_1; u_1 = c_1/l_1$.
STEP 2 For $i=2,\ldots,n-1$ set $l_i = a_i - b_iu_{i-1}; u_i = c_i/l_i$.
STEP 3 Set $l_n = a_n - b_nu_{n-1}$.
STEP 4 Set $z_1 = d_1/l_1$.
STEP 5 For $i=2,\ldots,n$ set $z_i = (d_i - b_iz_{i-1})/l_i$.
STEP 6 Set $x_n = z_n$.
STEP 7 For $i=n-1,\ldots,1$ set $x_i = z_i - u_ix_{i+1}$.
STEP 8 OUTPUT $(x_1,\ldots,x_n)$;
STOP.

28. First, $|l_{11}| = |a_{11}| > 0$ and $|u_{12}| = \frac{|a_{12}|}{|l_{11}|} < 1$. In general, assume $|l_{jj}| > 0$ and $|u_{j,j+1}| < 1$, for $j = 1, \ldots, i-1$. Then

$$|l_{ii}| = |a_{ii} - l_{i,i-1}u_{i-1,i}| = |a_{ii} - a_{i,i-1}u_{i-1,i}| \geq |a_{ii}| - |a_{i,i-1}u_{i-1,i}| > |a_{ii}| - |a_{i,i-1}| > 0,$$

and

$$|u_{i,i+1}| = \frac{|a_{i,i+1}|}{|l_{ii}|} < \frac{|a_{i,i+1}|}{|a_{ii}| - |a_{i,i-1}|} \leq 1,$$

for $i = 2, \ldots, n-1$. Further,

$$|l_{nn}| = |a_{nn} - l_{n,n-1}u_{n-1,n}| = |a_{nn} - a_{n,n-1}u_{n-1,n}| \geq |a_{nn}| - |a_{n,n-1}| > 0.$$

So

$$\det A = \det L \cdot \det U = l_{11} \cdot l_{22} \ldots l_{nn} \cdot 1 > 0.$$

29. $i_1 = 0.6785047, \quad i_2 = 0.4214953, \quad i_3 = 0.2570093, \quad i_4 = 0.1542056, \quad i_5 = 0.1028037$

30. The Crout Factorization Algorithm requires $5n - 4$ Multiplications/Divisions and $3n - 3$ Additions/Subtractions.

31. (a) Mating male i with female j produces offspring with the same wing characteristics as mating male j with female i.

    (b) No. Consider, for example, $\mathrm{x} = (1, 0, -1)^t$.

32. (a)

$$D^{1/2}D^{1/2} = \begin{bmatrix} \sqrt{d_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{d_{22}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{d_{nn}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{d_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{d_{22}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{d_{nn}} \end{bmatrix} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix} = D$$

(b) We have

$$\left(\hat{L}D^{1/2}\right)\left(\hat{L}D^{1/2}\right)^t = \hat{L}D^{1/2}\left(D^{1/2}\right)^t \hat{L}^t = \hat{L}D^{1/2}D^{1/2}\hat{L}^t = \hat{L}D\hat{L}^t = A.$$

Since $LL^t = A$, we have $\hat{L}D^{1/2} = L$.