1. (a) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 9 \\ -1 \\ 14 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & -9 & -4 \end{bmatrix}$

2. (a) $\begin{bmatrix} 4 \\ -18 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & 8 & -16 \end{bmatrix}$

3. (a) $\begin{bmatrix} -4 & 10 \\ 1 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 11 & 4 & -8 \\ 6 & 13 & -12 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 & -3 \\ 3 & 4 & -11 \\ -6 & -7 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 1 \\ -14 & 7 \\ 6 & 1 \end{bmatrix}$

4. (a) $\begin{bmatrix} -19 & 16 \\ -15 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -11 & 8 & 3 \\ -16 & 12 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & -16 & 15 \\ -14 & 24 & -18 \\ -23 & 14 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \\ 18 & -25 \end{bmatrix}$

5. Determine if the matrices are nonsingular, and if so, find the inverse.

   (a) The matrix is singular.

   (b) $\begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{bmatrix}$

   (c) The matrix is singular.

   (d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ -\frac{3}{14} & \frac{1}{7} & 0 & 0 \\ \frac{3}{28} & -\frac{11}{7} & 1 & 0 \\ -\frac{1}{2} & 1 & -1 & 1 \end{bmatrix}$

6. Determine if the matrices are nonsingular, and if so, find the inverse.

(a)

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{1}{10} & \frac{3}{5} & -\frac{1}{10} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{bmatrix}$$

(c) The matrix is singular.

(d)

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 1 \\ 3 & -1 & 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & \frac{5}{3} & \frac{5}{3} & -1 \\ -1 & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{4}{3} & 1 \end{bmatrix}$$

7. The solutions to the linear systems obtained in parts (a) and (b) are, from left to right,

$$3, -6, -2, -1 \quad \text{and} \quad 1, 1, 1, 1.$$

8. The solutions to the linear systems obtained in parts (a) and (b) are, from left to right and top to bottom:

$$-\frac{2}{7}, -\frac{13}{14}, -\frac{3}{14}; \quad \frac{17}{7}, -\frac{19}{14}, -\frac{41}{14}; \quad 1, 1, 1 \quad \text{and} \quad -\frac{1}{7}, \frac{2}{7}, \frac{1}{7}.$$

9. (a) Suppose $\tilde{A}$ and $\hat{A}$ are both inverses of A. Then $A\tilde{A} = \tilde{A}A = I$ and $A\hat{A} = \hat{A}A = I$. Thus,

$$\tilde{A} = \tilde{A}I = \tilde{A}(A\hat{A}) = (\tilde{A}A)\hat{A} = I\hat{A} = \hat{A}.$$

(b) We have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

So $(AB)^{-1} = B^{-1}A^{-1}$ since the inverse is unique.

(c) Since $A^{-1}A = AA^{-1} = I$, it follows that A^{-1} is nonsingular. Since the inverse is unique, we have $(A^{-1})^{-1} = A$.

10. (a) Not true. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. \quad \text{Then} \quad AB = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

is not symmetric.

(b) True. Let A be a nonsingular symmetric matrix. By Theorem 6.13 (d), $(A^{-1})^t = (A^t)^{-1}$. Thus, $(A^{-1})^t = (A^t)^{-1} = A^{-1}$ and A^{-1} is symmetric.

(c) Not true. Use the matrices A and B from part (a).

11. (a) If $C = AB$, where A and B are lower triangular, then $a_{ik} = 0$ if $k > i$ and $b_{kj} = 0$ if $k < j$. Thus

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=j}^{i} a_{ik} b_{kj},$$

which will have the sum zero unless $j \leq i$. Hence C is lower triangular.

(b) We have $a_{ik} = 0$ if $k < i$ and $b_{kj} = 0$ if $k > j$. The steps are similar to those in part (a).

(c) Let L be a nonsingular lower triangular matrix. To obtain the ith column of L^{-1}, solve n linear systems of the form

$$\begin{bmatrix} l_{11} & 0 & \cdots & \cdots & \cdots & 0 \\ l_{21} & l_{22} & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \ddots & & & \vdots \\ l_{i1} & l_{i2} & \cdots & l_{ii} & & \vdots \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & \cdots & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_i \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the 1 appears in the ith position to obtain the ith column of L^{-1}.

12. (a) Following the steps of Algorithm 6.1 with $m-1$ additional columns in the augmented matrix gives the following:

Reduction Steps 1–6:

Multiplications/Divisions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{1+(m+n-i)\} = \sum_{i=1}^{n-1}\{n(m+n+1)-(m+2n+1)i+i^2\}$$
$$= \frac{1}{2}mn^2 - \frac{1}{2}mn + \frac{1}{3}n^3 - \frac{1}{3}n$$

Additions/Subtractions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{m+n-i\} = \sum_{i=1}^{n-1}\{n(m+n)-(m+2n)i+i^2\}$$
$$= \frac{1}{2}mn^2 - \frac{1}{2}mn + \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$

Backward Substitution Steps 8–9:

Multiplications/Divisions:

$$m\left[1+\sum_{i=1}^{n-1}(n-i+1)\right] = m\left[1+\frac{n(n+1)}{2}-1\right] = \frac{1}{2}mn^2 + \frac{1}{2}mn$$

Additions/Subtractions:

$$m\left[\sum_{i=1}^{n-1}(n-i)\right] = \frac{1}{2}mn^2 - \frac{1}{2}mn$$

Total:

Multiplications/Divisions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n$

Additions/Subtractions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{2}n^2 - mn + \frac{1}{6}n$

(b) For the reduction phase: Multiplications/Divisions:

$$\sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}\left\{1+\sum_{k=i+1}^{n+m}1\right\} = \sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}(m+n+1-i) = \sum_{i=1}^{n}\{(n-1)(m+n+1)-(n-1)i\}$$
$$= \frac{1}{2}n^3 + mn^2 - mn - \frac{1}{2}n$$

Additions/Subtractions:

$$\sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}\sum_{k=i+1}^{n+m}1 = \sum_{i=1}^{n}\sum_{j=1 j\neq i}^{n}(n+m-i) = \sum_{i=1}^{n}\{(n-1)(m+n)-(n-1)i\}$$
$$= \frac{1}{2}n^3 + mn^2 - mn - n^2 + \frac{1}{2}n$$

Backward Substitution Steps:
Multiplications/Divisions:

$$\sum_{k=1}^{m}\sum_{i=1}^{n} 1 = mn$$

Additions/Subtractions: none
Totals:
Multiplications/Divisions: $\frac{1}{2}n^3 + mn^2 - \frac{1}{2}n$
Additions/Subtractions: $\frac{1}{2}n^3 + mn^2 - n^2 - mn + \frac{1}{2}n$

(c) When $m = n$ we have the following:

Gaussian Elimination
Multiplications/Divisions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n = \frac{4}{3}n^3 - \frac{1}{3}n$
Additions/Subtractions: $\frac{1}{3}n^3 + mn^2 - \frac{1}{2}n^2 - mn + \frac{1}{6}n = \frac{4}{3}n^3 - \frac{3}{2}n^2 + \frac{1}{6}n$

Gauss-Jordan Elimination
Multiplications/Divisions: $\frac{1}{2}n^3 + mn^2 - \frac{1}{2}n = \frac{3}{2}n^3 - \frac{1}{2}n$
Additions/Subtractions: $\frac{1}{2}n^3 + mn^2 - n^2 - mn + \frac{1}{2}n = \frac{3}{2}n^3 - 2n^2 + \frac{1}{2}n$

(d) To find the inverse of the $n \times n$ matrix A:
INPUT $n \times n$ matrix $A = (a_{ij})$.
OUTPUT $n \times n$ matrix $B = A^{-1}$.
Step 1 Initialize the $n \times n$ matrix $B = (b_{ij})$ to

$$b_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j \end{cases}$$

Step 2 For $i = 1, \ldots, n-1$ do Steps 3, 4, and 5.
Step 3 Let p be the smallest integer with $i \leq p \leq n$ and $a_{p,i} \neq 0$.
If no integer p can be found then
OUTPUT ('A is singular');
STOP.
Step 4 If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$.
Step 5 For $j = i+1, \ldots, n$ do Steps 6 through 9.
Step 6 Set $m_{ji} = a_{ji}/a_{ii}$.
Step 7 For $k = i+1, \ldots, n$
set $a_{jk} = a_{jk} - m_{ji}a_{ik}$; $a_{ij} = 0$.
Step 8 For $k = 1, \ldots, i-1$
set $b_{jk} = b_{jk} - m_{ji}b_{ik}$.
Step 9 Set $b_{ji} = -m_{ji}$.

Step 10 If $a_{nn} = 0$ then OUTPUT ('A is singular');
STOP.

Step 11 For $j = 1, \ldots, n$ do Steps 12, 13 and 14.

Step 12 Set $b_{nj} = b_{nj}/a_{nn}$.

Step 13 For $i = n-1, \ldots, j$
set $b_{ij} = \left(b_{ij} - \sum_{k=i+1}^{n} a_{ik}b_{kj}\right)/a_{ii}$.

Step 14 For $i = j-1, \ldots, 1$
set $b_{ij} = -\left[\sum_{k=i+1}^{n} a_{ik}b_{kj}\right]/a_{ii}$.

Step 15 OUTPUT (B);
STOP.

Reduction Steps 2–9:
Multiplications/Divisions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{1+\sum_{k=i+1}^{n}1+\sum_{k=1}^{i-1}1\right\} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{1+n-i+i-1\} = \frac{n^2(n-1)}{2}$$

Additions/Subtractions:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\sum_{k=i+1}^{n}1+\sum_{k=1}^{i-1}1\right\} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{n-i+i-1\} = \frac{n(n-1)^2}{2}$$

Backward Substitution Steps 11–14:
Multiplications/Divisions:

$$\sum_{j=1}^{n}\left\{1+\sum_{i=j}^{n-1}\left\{1+\sum_{k=i+1}^{n}1\right\}+\sum_{i=1}^{j-1}\left\{1+\sum_{k=i+1}^{n}1\right\}\right\} = \sum_{j=1}^{n}\left\{1+\sum_{i=j}^{n-1}(n+1-i)+\sum_{i=1}^{j-1}(n+1-i)\right\}$$

$$= \sum_{j=1}^{n}\left[1+\sum_{i=1}^{n-1}(n+1-i)\right]$$

$$= \sum_{j=1}^{n}\frac{n(n+1)}{2} = \frac{n^2(n+1)}{2}$$

Additions/Subtractions:

$$\sum_{j=1}^{n}\left\{\sum_{i=j}^{n-1}(1+n-i-1)+\sum_{i=1}^{j-1}(n-i-1)\right\} = \sum_{j=1}^{n}\sum_{i=1}^{n-1}(n-i)-j+1$$
$$= \sum_{j=1}^{n}\left[\frac{n(n-1)}{2}+1-j\right]$$
$$= \frac{n^2(n-1)}{2}+n-\frac{n(n+1)}{2}$$
$$= \frac{n^3}{2}-n^2+\frac{1}{2}n$$

Totals:

Multiplications/Divisions: $\frac{n^2(n-1)}{2}+\frac{n^2(n+1)}{2}=n^3$

Additions/Subtractions: $\frac{n(n-1)^2}{2}+\frac{n^3}{2}-n^2+\frac{1}{2}n=n^3-2n^2+n$

(e) Let $[A^{-1}]_{i,j}$ denote the entries of A^{-1}, for $1 \le i,j \le n$. For each $i = 1,\dots,n$, we have

$$x_i = \sum_{j=1}^{n}[A^{-1}]_{i,j}b_j.$$

This requires n multiplications and $n-1$ additions for each i. The total number of computations is:
n^2 Multiplications/Divisions and n^2-n Additions/Subtractions.

(f) For m linear systems, we have mn^2 Multiplications/Divisions and $m(n^2-n)$ Additions/Subtractions.

(g)

	Gaussian Elimination (part a)		Inverting A and forming $A^{-1}b$	
n	Multiplications Divisions	Additions Subtractions	Multiplications Divisions	Additions Subtractions
3	$9m+8$	$6m+5$	$9m+27$	$6m+12$
10	$100m+330$	$90m+285$	$100m+1000$	$90m+810$
50	$2500m+41650$	$2450m+40425$	$2500m+125000$	$2450m+120050$
100	$10000m+333300$	$9900m+328350$	$10000m+1000000$	$9900m+980100$

13. The answers are the same as those in Exercise 1.

14. No, since the products $A_{ij}B_{jk}$, for $1 \leq i,\ j,\ k \leq 2$, cannot be formed.

(c) The following are necessary and sufficient conditions:

(i) The number of columns of A is the same as the number of rows of B.

(ii) The number of vertical lines of A equals the number of horizontal lines of B.

(iii) The placement of the vertical lines of A is identical to placement of the horizontal lines of B.

15. (a) $A^2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ \frac{1}{6} & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^4 = A, \quad A^5 = A^2, \quad A^6 = I, \ldots$

(b)

	Year 1	Year 2	Year 3	Year 4
Age 1	6000	36000	12000	6000
Age 2	6000	3000	18000	6000
Age 3	6000	2000	1000	6000

(c) We have

$$A^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ \frac{1}{6} & 0 & 0 \end{bmatrix}.$$

The i, j-entry is the number of beetles of age i necessary to produce one beetle of age j.

16. (a) For each $k = 1, 2, \ldots, m$, the number a_{ik} represents the total number of plants of type v_i eaten by herbivores in the species h_k. The number of herbivores of types h_k eaten by species c_j is b_{kj}. Thus, the total number of plants of type v_i ending up in species c_j is $a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj} = (AB)_{ij}$.

(b) We first assume $n = m = k$ so that the matrices will have inverses. Let $x_1, \ldots, x_n$ represent the vegetations of type $v_1, \ldots, v_n$, let $y_1, \ldots, y_n$ represent the number of herbivores of species $h_1, \ldots, h_n$, and let $z_1, \ldots, z_n$ represent the number of carnivores of species $c_1, \ldots, c_n$.
If

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{then} \quad \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Thus, $(A^{-1})_{i,j}$ represents the amount of type v_j plants eaten by a herbivore of species h_i.
Similarly, if

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = B \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad \text{then} \quad \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = B^{-1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Thus, $(B^{-1})_{i,j}$ represents the number of herbivores of species h_j eaten by a carnivore of species c_i. If $x = Ay$ and $y = Bz$, then $x = ABz$ and $z = (AB)^{-1}x$. But, $y = A^{-1}x$ and $z = B^{-1}y$, so $z = B^{-1}A^{-1}x$.

17. (a) We have

$$\begin{bmatrix} 7 & 4 & 4 & 0 \\ -6 & -3 & -6 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2(x_0 - x_1) + \alpha_0 + \alpha_1 \\ 3(x_1 - x_0) - \alpha_1 - 2\alpha_0 \\ \alpha_0 \\ x_0 \end{bmatrix} = \begin{bmatrix} 2(x_0 - x_1) + 3\alpha_0 + 3\alpha_1 \\ 3(x_1 - x_0) - 3\alpha_1 - 6\alpha_0 \\ 3\alpha_0 \\ x_0 \end{bmatrix}$$

(b) $B = A^{-1} = \begin{bmatrix} -1 & -\frac{4}{3} & -\frac{4}{3} & 0 \\ 2 & \frac{7}{3} & 2 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

18. (a) In component form:

$$(a_{11}x_1 - b_{11}y_1 + a_{12}x_2 - b_{12}y_2) + (b_{11}x_1 + a_{11}y_1 + b_{12}x_2 + a_{12}y_2)i = c_1 + id_1,$$
$$(a_{21}x_1 - b_{21}y_1 + a_{22}x_2 - b_{22}y_2) + (b_{21}x_1 + a_{21}y_1 + b_{22}x_2 + a_{22}y_2)i = c_2 + id_2,$$

which yields

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 - b_{11}y_1 - b_{12}y_2 &= c_1, \\ b_{11}x_1 + b_{12}x_2 + a_{11}y_1 + a_{12}y_2 &= d_1, \\ a_{21}x_1 + a_{22}x_2 - b_{21}y_1 - b_{22}y_2 &= c_2, \\ b_{21}x_1 + b_{22}x_2 + a_{21}y_1 + a_{22}y_2 &= d_2. \end{aligned}$$

(b) The system

$$\begin{bmatrix} 1 & 3 & 2 & -2 \\ -2 & 2 & 1 & 3 \\ 2 & 4 & -1 & -3 \\ 1 & 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ -1 \end{bmatrix}$$

has the solution $x_1 = -1.2$, $x_2 = 1$, $y_1 = 0.6$, and $y_2 = -1$.