1. The Adams-Bashforth methods give the results in the following tables.

(a)

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	0.0268128	0.0268128	0.0268128	0.0268128	0.0268128
0.4	0.1200522	0.1507778	0.1507778	0.1507778	0.1507778
0.6	0.4153551	0.4613866	0.4960196	0.4960196	0.4960196
0.8	1.1462844	1.2512447	1.2961260	1.3308570	1.3308570
1.0	2.8241683	3.0360680	3.1461400	3.1854002	3.2190993

(b)

t	2-step	3-step	4-step	5-step	$y(t)$
2.2	1.3666667	1.3666667	1.3666667	1.3666667	1.3666667
2.4	1.6750000	1.6857143	1.6857143	1.6857143	1.6857143
2.6	1.9632431	1.9794407	1.9750000	1.9750000	1.9750000
2.8	2.2323184	2.2488759	2.2423065	2.2444444	2.2444444
3.0	2.4884512	2.5051340	2.4980306	2.5011406	2.5000000

(c)

t	2-step	3-step	4-step	5-step	$y(t)$
1.2	2.6187859	2.6187859	2.6187859	2.6187859	2.6187859
1.4	3.2734823	3.2710611	3.2710611	3.2710611	3.2710611
1.6	3.9567107	3.9514231	3.9520058	3.9520058	3.9520058
1.8	4.6647738	4.6569191	4.6582078	4.6580160	4.6580160
2.0	5.3949416	5.3848058	5.3866452	5.3862177	5.3862944

(d)

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	1.2529306	1.2529306	1.2529306	1.2529306	1.2529306
0.4	1.5986417	1.5712255	1.5712255	1.5712255	1.5712255
0.6	1.9386951	1.8827238	1.8750869	1.8750869	1.8750869
0.8	2.1766821	2.0844122	2.0698063	2.0789180	2.0789180
1.0	2.2369407	2.1115540	2.0998117	2.1180642	2.1179795

2. The Adams-Bashforth methods give the results in the following tables.

(a)

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	1.349857812				1.346153846
0.4	1.556590819	1.548505437	1.551742825		1.551724138
0.6	1.618098483	1.613103414	1.618495896	1.618045413	1.617647059
0.8	1.581123788	1.581537429	1.586784646	1.585534486	1.585365854
1.0	1.493132968	1.497482321	1.501365685	1.499907131	1.5

(b)

t	2-step	3-step	4-step	5-step	$y(t)$
1.2	−1.270097903				−1.268299404
1.4	−1.145585721	−1.14555416	−1.142395930		−1.142245242
1.6	−1.050436722	−1.045646519	−1.046822255	−1.046486051	−1.046559939
1.8	−0.9752223844	−0.9702763545	−0.9715163192	−0.9711385301	−0.9712326550
2.0	−0.9141608282	−0.9093053788	−0.9105163922	−0.9101442242	−0.9102392264

(c)

t	2-step	3-step	4-step	5-step	$y(t)$
1.4	−1.608147341				−1.555555556
1.8	−1.1429612993	−1.359101590	−1.404013030		−1.384615385
2.2	−1.331187984	−1.269164344	−1.314903737	−1.283404253	−1.294117647
2.6	−1.268712599	−1.217326056	−1.253881487	−1.229097787	−1.238095238
3.0	−1.225776866	−1.182688385	−1.212496957	−1.192208061	−1.200000000

(d)

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	1.058717655				1.057181007
0.4	1.207722688	1.202476610	1.200816932		1.201486010
0.6	1.389363032	1.381043540	1.379947852	1.381296110	1.380931216
0.8	1.562852894	1.554374654	1.554384988	1.555455875	1.555031423
1.0	1.707692156	1.700926322	1.701562220	1.702136018	1.701870053

3. The Adams-Bashforth methods give the results in the following tables.

(a)

t	2-step	3-step	4-step	5-step	$y(t)$
1.2	1.0161982	1.0149520	1.0149520	1.0149520	1.0149523
1.4	1.0497665	1.0468730	1.0477278	1.0475336	1.0475339
1.6	1.0910204	1.0875837	1.0887567	1.0883045	1.0884327
1.8	1.1363845	1.1327465	1.1340093	1.1334967	1.1336536
2.0	1.1840272	1.1803057	1.1815967	1.1810689	1.1812322

(b)

t	2-step	3-step	4-step	5-step	$y(t)$
1.4	0.4867550	0.4896842	0.4896842	0.4896842	0.4896817
1.8	1.1856931	1.1982110	1.1990422	1.1994320	1.1994386
2.2	2.1753785	2.2079987	2.2117448	2.2134792	2.2135018
2.6	3.5849181	3.6617484	3.6733266	3.6777236	3.6784753
3.0	5.6491203	5.8268008	5.8589944	5.8706101	5.8741000

(c)

t	2-step	3-step	4-step	5-step	$y(t)$
0.5	−1.5357010	−1.5381988	−1.5379372	−1.5378676	−1.5378828
1.0	−1.2374093	−1.2389605	−1.2383734	−1.2383693	−1.2384058
1.5	−1.0952910	−1.0950952	−1.0947925	−1.0948481	−1.0948517
2.0	−1.0366643	−1.0359996	−1.0359497	−1.0359760	−1.0359724

(d)

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	0.1739041	0.1627655	0.1627655	0.1627655	0.1626265
0.4	0.2144877	0.2026399	0.2066057	0.2052405	0.2051118
0.6	0.3822803	0.3747011	0.3787680	0.3765206	0.3765957
0.8	0.6491272	0.6452640	0.6487176	0.6471458	0.6461052
1.0	1.0037415	1.0020894	1.0064121	1.0073348	1.0022460

4. The Adams-Moulton methods give the results in the following tables.

(1a)

t_i	2-step	3-step	4-step	$y(t_i)$
0.2	0.0268128	0.0268128	0.0268128	0.0268128
0.4	0.1533627	0.1507778	0.1507778	0.1507778
0.6	0.5030068	0.4979042	0.4960196	0.4960196
0.8	1.3463142	1.3357923	1.3322919	1.3308570
1.0	3.2512866	3.2298092	3.2227484	3.2190993

(1c)

t_i	2-step	3-step	4-step	$y(t_i)$
1.2	2.6187859	2.6187859	2.6187859	2.6187859
1.4	3.2711394	3.2710611	3.2710611	3.2710611
1.6	3.9521454	3.9519886	3.9520058	3.9520058
1.8	4.6582064	4.6579866	4.6580211	4.6580160
2.0	5.3865293	5.3862558	5.3863027	5.3862944

(1d)

t_i	2-step	3-step	4-step	$y(t_i)$
0.2	1.2529306	1.2529306	1.2529306	1.2529306
0.4	1.5700866	1.5712255	1.5712255	1.5712255
0.6	1.8738414	1.8757546	1.8750869	1.8750869
0.8	2.0787117	2.0803067	2.0789471	2.0789180
1.0	2.1196912	2.1199024	2.1178679	2.1179795

5. Algorithm 5.4 gives the results in the following tables.

(a)

t_i	w_i	$y(t_i)$
0.2	0.0269059	0.0268128
0.4	0.1510468	0.1507778
0.6	0.4966479	0.4960196
0.8	1.3408657	1.3308570
1.0	3.2450881	3.2190993

(b)

t_i	w_i	$y(t_i)$
2.2	1.3666610	1.3666667
2.4	1.6857079	1.6857143
2.6	1.9749941	1.9750000
2.8	2.2446995	2.2444444
3.0	2.5003083	2.5000000

(c)

t_i	w_i	$y(t_i)$
1.2	2.6187787	2.6187859
1.4	3.2710491	3.2710611
1.6	3.9519900	3.9520058
1.8	4.6579968	4.6580160
2.0	5.3862715	5.3862944

(d)

t_i	w_i	$y(t_i)$
0.2	1.2529350	1.2529306
0.4	1.5712383	1.5712255
0.6	1.8751097	1.8750869
0.8	2.0796618	2.0789180
1.0	2.1192575	2.1179795

6. Algorithm 5.4 gives the results in the following tables.

(a)

t_i	w_i
0.2	1.3461536
0.4	1.5516984
0.6	1.6175240
0.8	1.5851977
1.0	1.4998499

(b)

t_i	w_i
1.2	−1.2682994
1.4	−1.1422321
1.6	−1.0465369
1.8	−0.9712077
2.0	−0.9102149

(c)

t_i	w_i
1.2	−1.7142452
1.6	−1.4545197
2.0	−1.3312918
2.4	−1.2614431
2.8	−1.2159883

(d)

t_i	w_i
0.2	1.0571822
0.4	1.2015654
0.6	1.3810423
0.8	1.5550968
1.0	1.7018941

7. The Adams Fourth-order Predictor-Corrector Algorithm gives the results in the following tables.

(a)

t	w	$y(t)$
1.2	1.0149520	1.0149523
1.4	1.0475227	1.0475339
1.6	1.0884141	1.0884327
1.8	1.1336331	1.1336536
2.0	1.1812112	1.1812322

(b)

t	w	$y(t)$
.4	0.4896842	0.4896817
1.8	1.1994245	1.1994386
2.2	2.2134701	2.2135018
2.6	3.6784144	3.6784753
3.0	5.8739518	5.8741000

(c)

t	w	$y(t)$
0.5	−1.5378788	−1.5378828
1.0	−1.2384134	−1.2384058
1.5	−1.0948609	−1.0948517
2.0	−1.0359757	−1.0359724

(d)

t	w	$y(t)$
0.2	0.1627655	0.1626265
0.4	0.2048557	0.2051118
0.6	0.3762804	0.3765957
0.8	0.6458949	0.6461052
1.0	1.0021372	1.0022460

8. The new algorithm gives the results in the following tables.

(a)

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
1.2	1.0149520	1.0149520	1.0149520	1.0149523
1.5	1.0672499	1.0672499	1.0672499	1.0672624
1.7	1.1106394	1.1106394	1.1106394	1.1106551
2.0	1.1812154	1.1812154	1.1812154	1.1812322

(b)

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
1.4	0.4896842	0.4896842	0.4896842	0.4896817
2.0	1.6613427	1.6613509	1.6613517	1.6612818
2.4	2.8767835	2.8768112	2.8768140	2.8765514
3.0	5.8754422	5.8756045	5.8756224	5.8741000

(c)

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
0.4	−1.6200494	−1.6200494	−1.6200494	−1.6200510
1.0	−1.2384104	−1.2384105	−1.2384105	−1.2384058
1.4	−1.1146533	−1.1146536	−1.1146536	−1.1146484
2.0	−1.0359139	−1.0359740	−1.0359740	−1.0359724

(d)

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
0.2	0.1627655	0.1627655	0.1627655	0.1626265
0.5	0.2774037	0.2773333	0.2773468	0.2773617
0.7	0.5000772	0.5000259	0.5000356	0.5000658
1.0	1.0022473	1.0022273	1.0022311	1.0022460

9. (a) With $h = 0.01$, the three-step Adams-Moulton method gives the values in the following table.

i	t_i	w_i
10	0.1	1.317218
20	0.2	1.784511

(b) Newton's method will reduce the number of iterations per step from three to two, using the stopping criterion

$$|w_i^{(k)} - w_i^{(k-1)}| \leq 10^{-6}.$$

10. Milne-Simpson's Predictor-Corrector method gives the results in the following tables.

(a)

i	t_i	w_i	$y(t_i)$
2	1.2	1.01495200	1.01495231
5	1.5	1.06725997	1.06726235
7	1.7	1.11065221	1.11065505
10	2.0	1.18122584	1.18123222

(b)

i	t_i	w_i	$y(t_i)$
2	1.4	0.48968417	0.48968166
5	2.0	1.66126150	1.66128176
7	2.4	2.87648763	2.87655142
10	3.0	5.87375555	5.87409998

(c)

i	t_i	w_i	$y(t_i)$
5	0.5	−1.53788255	−1.53788284
10	1.0	−1.23840789	−1.23840584
15	1.5	−1.09485532	−1.09485175
20	2.0	−1.03597247	−1.03597242

(d)

i	t_i	w_i	$y(t_i)$
2	0.2	0.16276546	0.16262648
5	0.5	0.27741080	0.27736167
7	0.7	0.50008713	0.50006579
10	1.0	1.00215439	1.00224598

11. (a) For some ξ_i in (t_{i-1}, t_i),

$$f(t, y(t_i)) = P_1(t) + \frac{f''(\xi_i, y(\xi_i))}{2}(t - t_i)(t - t_{i-1}),$$

where $P_1(t)$ is the linear Lagrange polynomial

$$P_1(t) = \frac{(t - t_{i-1})}{(t_i - t_{i-1})} f(t_i, y(t_i)) + \frac{(t - t_i)}{(t_{i-1} - t_i)} f(t_{i-1}, y(t_{i-1})).$$

Thus

$$\begin{aligned}\int_{t_i}^{t_{i+1}} P_1(t)\ dt =& \frac{f(t_i, y(t_i))}{t_i - t_{i-1}} \int_{t_i}^{t_{i+1}} (t - t_{i-1})\ dt + \frac{f(t_{i-1}, y(t_{i-1}))}{t_{i-1} - t_i} \int_{t_i}^{t_{i+1}} (t - t_i)\ dt \\ =& \frac{3h}{2} f(t_i, y(t_i)) - \frac{h}{2} f(t_{i-1}, y(t_{i-1})).\end{aligned}$$

Since $(t - t_i)(t - t_{i-1})$ does not change sign on (t_i, t_{i+1}), the Mean Value Theorem for Integrals gives

$$\begin{aligned}\int_{t_i}^{t_{i+1}} \frac{f''(\xi_i, y(\xi_i))(t - t_i)(t - t_{i-1})}{2}\ dt =& \frac{f''(\mu, y(\mu))}{2} \int_{t_i}^{t_{i+1}} (t - t_i)(t - t_{i-1})\ dt \\ =& \frac{5h^2 f''(\mu, y(\mu))}{12}.\end{aligned}$$

Replacing $y(t_j)$ with w_j, for $j = i - 1, i$, and $i + 1$ in the formula

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\ dt$$

gives

$$w_{i+1} = w_i + \frac{h\,[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})]}{2},$$

and the local truncation error is

$$\tau_{i+1}(h) = \frac{5h^2 y'''(\mu)}{12},$$

for some μ in (t_{i-1}, t_{i+1}).

(b) Using the backward difference polynomial with $m = 4$ gives

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,dt = \sum_{k=0}^{3} \nabla^k f(t_i, y(t_i)) h(-1)^k \int_0^1 \binom{-s}{k}\,ds$$
$$+ \frac{h^5}{24}\int_0^1 s(s+1)(s+2)(s+3) f^{(4)}(\xi_i, y(\xi_i))\,ds.$$

From Table 5.10 we have,

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,dt = h\Big[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))$$
$$+ \frac{3}{8}\nabla^3 f(t_i, y(t_i))\Big]$$
$$+ \frac{h^5}{24}\int_0^1 s(s+1)(s+2)(s+3) f^{(4)}(\xi_i, y(\xi_i))\,ds.$$

Since

$$\nabla f(t_i, y(t_i)) = f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1})),$$
$$\nabla^2 f(t_i, y(t_i)) = f(t_i, y(t_i)) - 2f(t_{i-1}, y(t_{i-1})) + f(t_{i-2}, y(t_{i-2})),$$
$$\nabla^3 f(t_i, y(t_i)) = f(t_i, y(t_i)) - 3f(t_{i-1}, y(t_{i-1})) + 3f(t_{i-2}, y(t_{i-2})) - f(t_{i-3}, y(t_{i-3})),$$

and $s(s+1)(s+2)(s+3)$ does not change sign on $(0, 1)$, we can simplify this and use the Mean Value Theorem for Integrals to obtain

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,dt = h\Big[55f(t_i, y(t_i)) - 59f(t_{i-1}, y(t_{i-1})) + 37f(t_{i-2}, y(t_{i-2}))$$
$$- 9f(t_{i-3}, y(t_{i-3}))\Big] + \frac{h^5}{24} f^{(4)}(\mu, y(\mu)) \int_0^1 s(s+1)(s+2)(s+3)\,ds,$$

for some μ in (t_{i-3}, t_{i+1}). The local truncation error form follows from the fact that

$$\int_0^1 s(s+1)(s+2)(s+3)\,ds = \frac{251}{30}.$$

12. Using the notation $y = y(t_i)$, $f = f(t_i, y(t_i))$, $f_t = f_t(t_i, y(t_i))$, etc., we have

$$y + hf + \frac{h^2}{2}(f_t + ff_y) + \frac{h^3}{6}\left(f_{tt} + f_t f_y + 2ff_{yt} + ff_y^2 + f^2 f_{yy}\right)$$
$$= y + ahf + bh\left[f - h(f_t + ff_y) + \frac{h^2}{2}\left(f_{tt} + f_t f_y + 2ff_{yt} + ff_y^2 + f^2 f_{yy}\right)\right]$$
$$+ ch\left[f - 2h(f_t + ff_y) + 2h^2\left(f_{tt} + f_t f_y + 2ff_{yt} + ff_y^2 + f^2 f_{yy}\right)\right].$$

Thus

$$a + b + c = 1, \quad -b - 2c = \frac{1}{2}, \quad \text{and} \quad \frac{1}{2}b + 2c = \frac{1}{6}.$$

This system has the solution

$$a = \frac{23}{12}, \quad b = -\frac{16}{12}, \quad \text{and} \quad c = \frac{5}{12}.$$

13. Newton's divided-difference formula gives

$$f(t,y(t)) = \frac{1}{2h^2}(t-t_i)(t-t_{i+1})f(t_{i-1},y(t_{i-1})) - \frac{1}{h^2}(t-t_{i-1})(t-t_{i+1})f(t_i,y(t_i)) + \frac{1}{2h^2}(t-t_{i-1})(t-t_i)f(t_{i+1},y(t_{i+1})) + \frac{1}{6}\frac{d^3}{dt^3}f(\xi,y(\xi))(t-t_{i-1})(t-t_i)(t-t_{i+1}),$$

so

$$\int_{t_i}^{t_{i+1}} y'(t)\ dt = \int_{t_i}^{t_{i+1}} f(t,y(t))\ dt$$

and

$$\begin{aligned} y(t_{i+1}) - y(t_i) =& \frac{f(t_{i-1},y(t_{i-1}))}{2h^2}\int_{t_i}^{t_{i+1}}(t-t_i)(t-t_{i+1})\ dt - \frac{f(t_i,y(t_i))}{h^2}\int_{t_i}^{t_{i+1}}(t-t_{i-1})(t-t_i)\ dt \\ &+ \frac{f(t_{i+1},y(t_{i+1}))}{2h^2}\int_{t_i}^{t_{i+1}}(t-t_{i-1})(t-t_i)\ dt + \int_{t_i}^{t_{i+1}}\frac{f'''(\xi,y(\xi))}{6}(t-t_{i-1})(t-t_i)(t-t_{i+1})\ dt \\ =& \frac{-h}{12}f(t_{i-1},y(t_{i-1})) + \frac{2h}{3}f(t_i,y(t_i)) + \frac{5h}{12}f(t_{i+1},y(t_{i+1})) \\ &+ \frac{f'''(\mu,y(\mu))}{6}\int_{t_i}^{t_{i+1}}(t-t_{i-1})(t-t_i)(t-t_{i+1})\ dt. \end{aligned}$$

The last part follows from Theorem 4.2. Further integration yields Formula (5.37) and the local truncation error.

14. We have

$$\begin{aligned} y(t_{i+1}) - y(t_{i-1}) &= \int_{t_{i-1}}^{t_{i+1}} f(t,y(t))dt \\ &= \frac{h}{3}\left[f(t_{i-1},y(t_{i-1})) + 4f(t_i,y(t_i)) + f(t_{i+1},y(t_{i+1}))\right] - \frac{h^5}{90}f^{(4)}(\xi,y(\xi)). \end{aligned}$$

This leads to the difference equation

$$w_{i+1} = w_{i-1} + \frac{h\left[f(t_{i-1},w_{i-1}) + 4f(t_i,w_i) + f(t_{i+1},w_{i+1})\right]}{3},$$

with local truncation error

$$\tau_{i+1}(h) = \frac{-h^4 y^{(5)}(\xi)}{90}.$$

15. To derive Milne's method, integrate $y'(t) = f(t, y(t))$ on the interval $[t_{i-3}, t_{i+1}]$ to obtain

$$y(t_{i+1}) - y(t_{i-3}) = \int_{t_{i-3}}^{t_{i+1}} f(t, y(t))\, dt.$$

Using the open Newton-Cotes formula (4.31) on page 201, we have

$$y(t_{i+1}) - y(t_{i-3}) = \frac{4h[2f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1})) + 2f(t_{i-2}, y(t_{i-2}))]}{3} + \frac{14h^5 f^{(4)}(\xi, y(\xi))}{45}.$$

The difference equation becomes

$$w_{i+1} = w_{i-3} + \frac{h[8f(t_i, w_i) - 4f(t_{i-1}, w_{i-1}) + 8f(t_{i-2}, w_{i-2})]}{3},$$

with local truncation error

$$\tau_{i+1}(h) = \frac{14h^4 y^{(5)}(\xi)}{45}.$$

16. The entries are generated by evaluating the following integrals:

$$\begin{aligned}
k = 0 &: (-1)^k \int_0^1 \binom{-s}{k}\, ds = \int_0^1 ds = 1,\\
k = 1 &: (-1)^k \int_0^1 \binom{-s}{k}\, ds = -\int_0^1 -s\, ds = \frac{1}{2},\\
k = 2 &: (-1)^k \int_0^1 \binom{-s}{k}\, ds = \int_0^1 \frac{s(s+1)}{2}\, ds = \frac{5}{12},\\
k = 3 &: (-1)^k \int_0^1 \binom{-s}{k}\, ds = -\int_0^1 \frac{-s(s+1)(s+2)}{6}\, ds = \frac{3}{8},\\
k = 4 &: (-1)^k \int_0^1 \binom{-s}{k}\, ds = \int_0^1 \frac{s(s+1)(s+2)(s+3)}{24}\, ds = \frac{251}{720}, \quad \text{and}\\
k = 5 &: (-1)^k \int_0^1 \binom{-s}{k}\, ds = -\int_0^1 -\frac{s(s+1)(s+2)(s+3)(s+4)}{120}\, ds = \frac{95}{288}.
\end{aligned}$$