1. Euler's method gives the results in the following tables.

(a)

t_i	w_i	y_i
0.200	0.027182818	0.449328964
0.500	0.000027183	0.030197383
0.700	0.000000272	0.004991594
1.000	0.000000000	0.000335463

(b)

t_i	w_i	y_i
0.200	0.373333333	0.046105213
0.500	−0.093333333	0.250015133
0.700	0.146666667	0.490000277
1.000	1.333333333	1.000000001

(c)

t_i	w_i	y_i
0.500	16.47925	0.479470939
1.000	256.7930	0.841470987
1.500	4096.142	0.997494987
2.000	65523.12	0.909297427

(d)

t_i	w_i	y_i
0.200	6.128259	1.000000001
0.500	−378.2574	1.000000000
0.700	−6052.063	1.000000000
1.000	387332.0	1.000000000

2. Euler's method gives the results in the following tables.

(a)

t_i	w_i	y_i
0.2	1.4631026	1.5892822
0.4	1.5421118	1.6271500
0.6	1.8223081	1.8719059
0.8	2.2104643	2.2438566
1.0	2.6960402	2.7250198

(b)

t_i	w_i	y_i
0.2	0.2	0.56787944
0.4	0.4	0.44978707
0.6	0.6	0.60673795
0.8	0.8	0.80091188
1.0	1.0	1.00012341

(c)

t_i	w_i	y_i
1.25	0.2500008	0.48848225
1.75	−1.5816058	0.18657591
2.25	−13.2433087	0.08779149
2.75	−100.7565966	0.04808415

(d)

t_i	w_i	y_i
0.25	5.0000000	0.96217447
0.75	65.137212	0.73168856
1.25	1030.7344	0.31532236
1.75	16486.497	−0.17824606

3. The Runge-Kutta fourth order method gives the results in the following tables.

(a)

t_i	w_i	y_i
0.200	0.45881186	0.44932896
0.500	0.03181595	0.03019738
0.700	0.00537013	0.00499159
1.000	0.00037239	0.00033546

(b)

t_i	w_i	y_i
0.200	0.07925926	0.04610521
0.500	0.25386145	0.25001513
0.700	0.49265127	0.49000028
1.000	1.00250560	1.00000000

(c)

t_i	w_i	y_i
0.500	188.3082	0.47947094
1.000	35296.68	0.84147099
1.500	6632737	0.99749499
2.000	1246413200	0.90929743

(d)

t_i	w_i	y_i
0.200	−215.7459	1.00000000
0.500	−555750.0	1.00000000
0.700	−104435653	1.00000000
1.000	−269031268010	1.00000000

4. The Runge-Kutta fourth order method gives the results in the following tables.

(a)

t_i	w_i	y_i
0.2	1.5895980	1.5892822
0.4	1.6274132	1.6271500
0.6	1.8720749	1.8719059
0.8	2.2439777	2.2438566
1.0	2.7251239	2.7250198

(b)

t_i	w_i	y_i
0.2	0.5822584	0.56787944
0.4	0.4537551	0.44978707
0.6	0.6075593	0.60673795
0.8	0.8010630	0.80091188
1.0	1.0001495	1.00012341

(c)

t_i	w_i	y_i
1.25	0.8240614	0.48848225
1.75	4.9786559	0.18657591
2.25	66.9187958	0.08779149
2.75	930.8525134	0.04808415

(d)

t_i	w_i	y_i
0.25	−12.8205769	0.96217447
0.75	−2591.69792	0.73168856
1.25	−487165.725	0.31532236
1.75	−91547464.8	−0.17824606

5. The Adams Fourth-Order Predictor-Corrector Algorithm gives the results in the following tables.

(a)

t_i	w_i	y_i
0.200	0.4588119	0.4493290
0.500	−0.0112813	0.0301974
0.700	0.0013734	0.0049916
1.000	0.0023604	0.0003355

(b)

t_i	w_i	y_i
0.200	0.0792593	0.0461052
0.500	0.1554027	0.2500151
0.700	0.5507445	0.4900003
1.000	0.7278557	1.0000000

(c)

t_i	w_i	y_i
0.500	188.3082	0.4794709
1.000	38932.03	0.8414710
1.500	9073607	0.9974950
2.000	2115741299	0.9092974

(d)

t_i	w_i	y_i
0.200	−215.7459	1.000000001
0.500	−682637.0	1.000000000
0.700	−159172736	1.000000000
1.000	−566751172258	1.000000000

6. The Adams Fourth-Order Predictor-Corrector Algorithm gives the results in the following tables.

(a)

t_i	w_i	$y(t_i)$
0.2	1.5895980	1.5892822
0.4	1.6263206	1.6271500
0.6	1.8709345	1.8719059
0.8	2.2432194	2.2438566
1.0	2.7246942	2.7250198

(b)

t_i	w_i	$y(t_i)$
0.2	0.5822584	0.56787944
0.4	0.3911238	0.44978707
0.6	0.5971191	0.60673795
0.8	0.7826888	0.80091188
1.0	1.0082013	1.00012341

(c)

t_i	w_i	$y(t_i)$
1.25	0.8240614	0.48848225
1.75	4.9786559	0.18657591
2.25	200.8236197	0.08779149
2.75	10272.0539300	0.04808415

(d)

t_i	w_i	$y(t_i)$
0.25	−12.820577	0.96217447
0.75	−2591.6979	0.73168856
1.25	−598288.88	0.31532236
1.75	−139504990	−0.17824606

7. The Trapezoidal Algorithm gives the results in the following tables.

(a)

t_i	w_i	k	y_i
0.200	0.39109643	2	0.44932896
0.500	0.02134361	2	0.03019738
0.700	0.00307084	2	0.00499159
1.000	0.00016759	2	0.00033546

(b)

t_i	w_i	k	y_i
0.200	0.04000000	2	0.04610521
0.500	0.25000000	2	0.25001513
0.700	0.49000000	2	0.49000028
1.000	1.00000000	2	1.00000000

(c)

t_i	w_i	k	y_i
0.500	0.66291133	2	0.47947094
1.000	0.87506346	2	0.84147099
1.500	1.00366141	2	0.99749499
2.000	0.91053267	2	0.90929743

(d)

t_i	w_i	k	y_i
0.200	−1.07568307	4	1.00000000
0.500	−0.97868360	4	1.00000000
0.700	−0.99046408	3	1.00000000
1.000	−1.00284456	3	1.00000000

8. The Trapezoidal Algorithm gives the results in the following tables.

(a)

t_i	w_i	k	$y(t_i)$
0.2	1.58152229	2	1.5892822
0.4	1.62161373	2	1.6271500
0.6	1.86902118	2	1.8719059
0.8	2.24264359	2	2.2438566
1.0	2.72470482	2	2.7250198

(b)

t_i	w_i	k	$y(t_i)$
0.2	0.50203131	2	0.56787944
0.4	0.43355903	2	0.44978707
0.6	0.60372878	2	0.60673795
0.8	0.80041431	2	0.80091188
1.0	1.00004603	2	1.00012341

(c)

t_i	w_i	k	$y(t_i)$
1.25	0.49787835	2	0.48848225
1.75	0.18503499	2	0.18657591
2.25	0.08754204	2	0.08779149
2.75	0.04802381	2	0.04808415

(d)

t_i	w_i	k	$y(t_i)$
0.25	1.39753016	2	0.96217447
0.75	0.81057360	2	0.73168856
1.25	0.33002435	2	0.31532236
1.75	-0.17533094	2	−0.17824606

9. The Runge-Kutta fourth-order method gives the results in the following tables.

(a)

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.100	−96.33011	0.66987648	193.6651	−0.33491554
0.200	−28226.32	0.67915383	56453.66	−0.33957692
0.300	−8214056	0.69387881	16428113	−0.34693941
0.400	−2390290586	0.71354670	4780581173	−0.35677335
0.500	−695574560790	0.73768711	1391149121600	−0.36884355

(b)

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.100	0.61095960	0.66987648	−0.21708179	−0.33491554
0.200	0.66873489	0.67915383	−0.31873903	−0.33957692
0.300	0.69203679	0.69387881	−0.34325535	−0.34693941
0.400	0.71322103	0.71354670	−0.35612202	−0.35677335
0.500	0.73762953	0.73768711	−0.36872840	−0.36884355

10. Since $y' = \lambda y$, we have

$$k_1 = h\lambda w_i, \quad k_2 = h\lambda(w_i + h\lambda w_i/2), \quad k_3 = h\lambda\left(w_i + h\lambda w_i/2 + h^2\lambda^2 w_i/4\right),$$

and

$$k_4 = h\lambda\left(w_i + h\lambda w_i + h^2\lambda^2 w_i/2 + h^3\lambda^3 w_i/4\right).$$

Thus

$$\begin{aligned} w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ &= w_i + \frac{w_i}{6}\left(h\lambda + 2h\lambda + h^2\lambda^2 + 2h\lambda + h^2\lambda^2 + h^3\lambda^3/2 + h\lambda + h^2\lambda^2 + \frac{1}{2}h^3\lambda^3 + \frac{1}{4}h^4\lambda^4\right) \\ &= \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right) w_i. \end{aligned}$$

11. Using Eq. (4.25) gives $\tau_{i+1} = \frac{-y'''(\xi_i)}{12}h^2$, for some $t_i < \xi_i < t_{i+1}$, and by Definition 5.18, the Trapezoidal method is consistent. Once again using Eq. (4.25) gives

$$y(t_{i+1}) = y(t_i) + \frac{h}{2}[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))] - \frac{y'''(\xi_i)}{12}h^3.$$

Subtracting the difference equation and using the Lipschitz constant L for f gives

$$|y(t_{i+1}) - w_{i+1}| \le |y(t_i) - w_i| + \frac{hL}{2}|y(t_i) - w_i| + \frac{hL}{2}|y(t_{i+1}) - w_{i+1}| + \frac{h^3}{12}|y'''(\xi_i)|.$$

Let $M = \max_{a \le x \le b} |y'''(x)|$. Then, assuming $hL \ne 2$,

$$|y(t_{i+1}) - w_{i+1}| \le \frac{2 + hL}{2 - hL}|y(t_i) - w_i| + \frac{h^3}{6(2 - hL)}M.$$

Using Lemma 5.8, we have

$$|y(t_{i+1}) - w_{i+1}| \le e^{2(b-a)L/(2-hL)}\left[\frac{Mh^2}{12L} + |\alpha - w_0|\right] - \frac{Mh^2}{12L}.$$

So if $hL \ne 2$, the Trapezoidal method is convergent, and consequently stable.

12. The Backward Euler method applied to $y' = \lambda y$ gives

$$w_{i+1} = \frac{w_i}{1 - h\lambda}, \quad \text{so} \quad Q(h\lambda) = \frac{1}{1 - h\lambda}.$$

13. The following tables list the results of the Backward Euler method applied to the problems in Exercise 1.

(1a)

i	t_i	w_i	k	y_i
2	0.2	0.75298666	2	0.44932896
5	0.5	0.10978082	2	0.03019738
7	0.7	0.03041020	2	0.00499159
10	1.0	0.00443362	2	0.00033546

(1b)

i	t_i	w_i	k	y_i
2	0.2	0.08148148	2	0.04610521
5	0.5	0.25635117	2	0.25001513
7	0.7	0.49515013	2	0.49000028
10	1.0	1.00500556	2	1.00000000

(1c)

i	t_i	w_i	k	y_i
2	0.5	0.50495522	2	0.47947094
4	1.0	0.83751817	2	0.84147099
6	1.5	0.99145076	2	0.99749499
8	2.0	0.90337560	2	0.90929743

(1d)

i	t_i	w_i	k	y_i
2	0.2	1.00348713	3	1.00000000
5	0.5	1.00000262	2	1.00000000
7	0.7	1.00000002	1	1.00000000
10	1.0	1.00000000	1	1.00000000

14. The following tables list the results of the Backward Euler method applied to the problems in Exercise 2.

(2a)

i	t_i	w_i	k	y_i
2	0.2	1.67216224	2	1.58928220
4	0.4	1.69987544	2	1.62715998
6	0.6	1.92400672	2	1.87190587
8	0.8	2.28233119	2	2.24385657
10	1.0	2.75757631	2	2.72501978

(2b)

i	t_i	w_i	k	y_i
2	0.2	0.87957046	2	0.56787944
4	0.4	0.56989261	2	0.44978707
6	0.6	0.64247315	2	0.60673795
8	0.8	0.81061829	2	0.80091188
10	1.0	1.00265457	2	1.00012341

(2c)

i	t_i	w_i	k	y_i
1	1.25	0.55006309	2	0.51199999
3	1.75	0.19753128	2	0.18658892
5	2.25	0.09060118	2	0.08779150
7	2.75	0.04900207	2	0.04808415

(2d)

i	t_i	w_i	k	y_i
1	0.25	0.79711852	2	0.96217447
3	0.75	0.72203841	2	0.73168856
5	1.25	0.31248267	2	0.31532236
7	1.75	-0.17796016	2	-0.17824606

15. (a) The Trapezoidal method applied to the test equation gives

$$w_{j+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}\, w_j, \quad \text{so} \quad Q(h\lambda) = \frac{2 + h\lambda}{2 - h\lambda}.$$

Thus, $|Q(h\lambda)| < 1$, whenever $\mathrm{Re}(h\lambda) < 0$.

(b) The Backward Euler method applied to the test equation gives

$$w_{j+1} = \frac{w_j}{1 - h\lambda}, \quad \text{so} \quad Q(h\lambda) = \frac{1}{1 - h\lambda}.$$

Thus, $|Q(h\lambda)| < 1$, whenever $\mathrm{Re}(h\lambda) < 0$.