1. Let L be the Lipschitz constant for ϕ. Then

$$u_{i+1} - v_{i+1} = u_i - v_i + h[\phi(t_i, u_i, h) - \phi(t_i, v_i, h)],$$

so

$$|u_{i+1} - v_{i+1}| \le (1 + hL)|u_i - v_i| \le (1 + hL)^{i+1}|u_0 - v_0|.$$

2. (a) For the Adams-Bashforth Method,

$$F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}, w_{i-3}) = \frac{1}{24}\Big[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})\Big],$$

so if $f \equiv 0$, then $F \equiv 0$. The same result holds for the Adams-Moulton method.

(b) If f has Lipschitz constant L, then

$$|F(t_i, h, w_{i+1}, \ldots, w_{i-3}) - F(t_i, h, v_{i+1}, \ldots, v_{i-3})| \le \frac{L}{24}\Big[55|w_i - v_i| + 59|w_{i-1} - v_{i-1}| + 37|w_{i-2} - v_{i-2}| + 9|w_{i-3} - v_{i-3}|\Big],$$

so $C = \frac{59}{24}L$ will suffice. A similar result holds for the Adams-Moulton method, but with $C = \frac{19}{24}L$.

3. By Exercise 31 in Section 5.4, we have

$$\begin{aligned}\phi(t, w, h) &= \frac{1}{6}f(t, w) + \frac{1}{3}f\left(t + \frac{1}{2}h, w + \frac{1}{2}hf(t, w)\right)\\ &+ \frac{1}{3}f\left(t + \frac{1}{2}h, w + \frac{1}{2}hf\left(t + \frac{1}{2}h, w + \frac{1}{2}hf(t, w)\right)\right)\\ &+ \frac{1}{6}f\left(t + h, w + hf\left(t + \frac{1}{2}h, w + \frac{1}{2}hf\left(t + \frac{1}{2}h, w + \frac{1}{2}hf(t, w)\right)\right)\right),\end{aligned}$$

so

$$\phi(t, w, 0) = \frac{1}{6}f(t, w) + \frac{1}{3}f(t, w) + \frac{1}{3}f(t, w) + \frac{1}{6}f(t, w) = f(t, w).$$

4. (a) Expand $y(t_{i+1})$ and $y(t_{i+2})$ in Taylor polynomials and simplify.

(b) $w_2 = 0.18065 \approx y(0.2) = 0.18127$,$w_5 = 0.35785 \approx y(0.5) = 0.39347$, $w_7 = 0.15340 \approx y(0.7) = 0.50341$, and $w_{10} = -9.7822 \approx y(1.0) = 0.63212$

(c) $w_{20} = -60.402 \approx y(0.2)$, $w_{50} = -1.37 \times 10^{17} \approx y(0.5)$, $w_{70} = -5.11 \times 10^{26} \approx y(0.7)$, and $w_{100} = -1.16 \times 10^{41} \approx y(1.0)$

(d) The method is consistent but not stable or convergent.

5. (a) The local truncation error is $\tau_{i+1} = \frac{1}{4}h^3y^{(4)}(\xi_i)$, for some ξ, where $t_{i-2} < \xi_i < t_{i+1}$.

(b) The method is consistent but unstable and not convergent.

6. For $h = 0.1$:
$w_{10} = 0.3678826 \approx y(1) = 0.3678794$, and $w_{100} = 3.84917 \approx y(10) = 0.0000454$.
For $h = 0.01$:
$w_{100} = 0.3678794 \approx y(1) = 0.3678794$ and $w_{1000} = 0.0001091 \approx y(10) = 0.0000454$.

7. The method is unstable.

8. $w_2 = 4\varepsilon$, $w_3 = 13\varepsilon$, $w_4 = 40\varepsilon$, $w_5 = 121\varepsilon$, $w_6 = 364\varepsilon$