1. (a) Since $f(t,y) = y\cos t$, we have $\dfrac{\partial f}{\partial y}(t,y) = \cos t$, and f satisfies a Lipschitz condition in y with $L = 1$ on

$$D = \{(t,y)|0 \le t \le 1, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is $y(t) = e^{\sin t}$.

(b) Since $f(t,y) = \dfrac{2}{t}y + t^2e^t$, we have $\dfrac{\partial f}{\partial y} = \dfrac{2}{t}$, and f satisfies a Lipschitz condition in y with $L = 2$ on

$$D = \{(t,y)|1 \le t \le 2, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is $y(t) = t^2(e^t - e)$.

(c) Since $f(t,y) = -\dfrac{2}{t}y + t^2e^t$, we have $\dfrac{\partial f}{\partial y} = -\dfrac{2}{t}$, and f satisfies a Lipschitz condition in y with $L = 2$ on

$$D = \{(t,y)|1 \le t \le 2, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is

$$y(t) = (t^4e^t - 4t^3e^t + 12t^2e^t - 24te^t + 24e^t + (\sqrt{2} - 9)e)/t^2.$$

(d) Since $f(t,y) = \dfrac{4t^3y}{1+t^4}$, we have $\dfrac{\partial f}{\partial y} = \dfrac{4t^3}{1+t^4}$, and f satisfies a Lipschitz condition in y with $L = 2$ on

$$D = \{(t,y)|0 \le t \le 1, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is $y(t) = 1 + t^4$.

2. (a) Since $f(t,y) = e^{t-y}$, we have $\dfrac{\partial f}{\partial y}(t,y) = -e^{t-y}$, and f does not satisfies a Lipschitz condition in y on
$$D = \{(t,y)|0 \le t \le 1, -\infty < y < \infty\}.$$
But there is a unique solution, which is $y(t) = \ln(e^t - 1 + e)$.

(b) Since $f(t,y) = t^{-2}(\sin(2t) - 2ty)$, we have $\dfrac{\partial f}{\partial y} = -2/t$, and f satisfies a Lipschitz condition in y with $L = 2$ on
$$D = \{(t,y)|1 \le t \le 2, -\infty < y < \infty\}.$$
Also, f is continuous on D, so there exists a unique solution, which is $y(t) = \dfrac{1}{2}(4 + \cos 2 - \cos(2t))t^{-2}$.

(c) Since $f(t,y) = -y + ty^{1/2}$, we have $\dfrac{\partial f}{\partial y} = -1 + (t/2)y^{-1/2}$, and f does not satisfies a Lipschitz condition in y on
$$D = \{(t,y)|2 \le t \le 3, -\infty < y < \infty\}.$$
But there is a unique solution, which is $y(t) = \left(t - 2 + \sqrt{2}e^{1-t/2}\right)^2$.

(d) Since $f(t,y) = \dfrac{ty + y}{ty + t}$, we have $\dfrac{\partial f}{\partial y} = \dfrac{t+1}{t(y+1)^2}$, and f does not satisfies a Lipschitz condition in y on
$$D = \{(t,y)|2 \le 4 \le 1, -\infty < y < \infty\}.$$
But there is a unique solution, which is implicitly given by $y(t) - t - 2 = \ln(2t/y(t))$.

3. (a) Lipschitz constant $L = 1$; it is a well-posed problem.
(b) Lipschitz constant $L = 1$; it is a well-posed problem.
(c) Lipschitz constant $L = 1$; it is a well-posed problem.
(d) The function f does not satisfy a Lipschitz condition, so Theorem 5.6 cannot be used.

4. (a) The function f does not satisfy a Lipschitz condition, so Theorem 5.6 cannot be used.
(b) Lipschitz constant $L = 1$; it is a well-posed problem.
(c) Lipschitz constant $L = 1$; it is a well-posed problem.
(d) The function f does not satisfy a Lipschitz condition, so Theorem 5.6 cannot be used.

5. (a) Differentiating

$$y^3t + yt = 2 \quad \text{gives} \quad 3y^2y't + y^3 + y't + y = 0.$$

Solving for y' gives the original differential equation, and setting $t = 1$ and $y = 1$ verifies the initial condition. To approximate $y(2)$, use Newton's method to solve the equation $y^3 + y - 1 = 0$. This gives $y(2) \approx 0.6823278$.

(b) Differentiating

$$y \sin t + t^2e^y + 2y - 1 = 0 \quad \text{gives} \quad y' \sin t + y \cos t + 2te^y + t^2e^yy' + 2y' = 0.$$

Solving for y' gives the original differential equation, and setting $t = 1$ and $y = 0$ verifies the initial condition. To approximate $y(2)$, use Newton's method to solve the equation $(2 + \sin 2)y + 4e^y - 1 = 0$. This gives $y(2) \approx -0.4946599$.

6. Let (t, y_1) and (t, y_2) be in D. Holding t fixed, define $g(y) = f(t, y)$. Suppose $y_1 < y_2$. Since the line joining (t, y_1) to (t, y_2) lies in D and f is continuous on D, we have $g \in C[y_1, y_2]$.

Further, $g'(y) = \dfrac{\partial f(t, y)}{\partial y}$. Using the Mean Value Theorem on g, a number ξ, for $y_1 < \xi < y_2$, exists with

$$g(y_2) - g(y_1) = g'(\xi)(y_2 - y_1).$$

Thus

$$f(t, y_2) - f(t, y_1) = \frac{\partial f(t, \xi)}{\partial y}(y_2 - y_1)$$

and

$$|f(t, y_2) - f(t, y_1)| \le L\,|y_2 - y_1|\,.$$

The proof is similar if $y_2 < y_1$. Therefore, f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

7. Let (t_1, y_1) and (t_2, y_2) be in D, with $a \le t_1 \le b$, $a \le t_2 \le b$, $-\infty < y_1 < \infty$, and $-\infty < y_2 < \infty$.

For $0 \le \lambda \le 1$, we have

$$(1 - \lambda)a \le (1 - \lambda)t_1 \le (1 - \lambda)b \quad \text{and} \quad \lambda a \le \lambda t_2 \le \lambda b.$$

Hence

$$a = (1 - \lambda)a + \lambda a \le (1 - \lambda)t_1 + \lambda t_2 \le (1 - \lambda)b + \lambda b = b.$$

Also, $-\infty < (1 - \lambda)y_1 + \lambda y_2 < \infty$, so D is convex.

8. (a) Since $y(t) = 1 - e^{-t}$, we have

$$z(t) = 1 - e^{-t} + \delta\left(t - 1 + e^{-t}\right) + \varepsilon_0 e^{-t} \quad \text{and} \quad |y(t) - z(t)| \leq 2|\delta| + |\varepsilon_0| < 3\varepsilon,$$

so the problem is well posed.

(b) Since $y(t) = -t - 1$, we have

$$z(t) = -t - 1 + \delta\left(-t - 1 + e^{t}\right) + \varepsilon_0 e^{t} \quad \text{and} \quad |y(t) - z(t)| \leq 4.4|\delta| + 7.4|\varepsilon_0| < 11.8\varepsilon,$$

so the problem is well posed.

(c) Since $y(t) = t^2\left(e^t - e\right)$, we have

$$z(t) = t^2\left(e^t - e\right) + t^2\left(\varepsilon_0 + \delta \ln t\right) \quad \text{and} \quad |y(t) - z(t)| \leq 4(|\varepsilon_0| + \ln 2|\delta|) < 6.8\varepsilon,$$

so the problem is well posed.

(d) Since

$$y(t) = \frac{t^4e^t - 4t^3e^t + 12t^2e^t - 24te^t + 24e^t}{t^2} + \frac{\left(\sqrt{2} - 9\right)e}{t^2},$$

we have

$$z(t) = \frac{t^4e^t - 4t^3e^t + 12t^2e^t - 24te^t + 24e^t}{t^2} + \frac{\left(\sqrt{2} - 9\right)e}{t^2} + \frac{1}{4}\delta t^2 + \frac{\varepsilon_0 - \delta/4}{t^2}$$

and

$$|y(t) - z(t)| \leq |\delta| + |\varepsilon_0| + |\delta|/4 < 2.25\varepsilon,$$

so the problem is well posed.

9. (a) Since $y' = f(t, y(t))$, we have

$$\int_a^t y'(z)\, dz = \int_a^t f(z, y(z))\, dz.$$

So

$$y(t) - y(a) = \int_a^t f(z, y(z))\, dz$$

and

$$y(t) = \alpha + \int_a^t f(z, y(z))\, dz.$$

The iterative method follows from this equation.

(b) We have

$$y_0(t) = 1,\ y_1(t) = 1 + \frac{1}{2}t^2,\ y_2(t) = 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3, \text{ and } y_3(t) = 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4.$$

(c) We have

$$y(t) = 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \cdots.$$