1. (a) $f'(1) \approx 1.0000109$
   (b) $f'(0) \approx 2.0000000$
   (c) $f'(1.05) \approx 2.2751459$
   (d) $f'(2.3) \approx -19.646799$

2. (a) $f'(1) \approx 0.99999998$
   (b) $f'(0) \approx 1.9999999$
   (c) $f'(1.05) \approx 2.2751458$
   (d) $f'(2.3) \approx -19.646796$

3. (a) $f'(1) \approx 1.001$
   (b) $f'(0) \approx 1.999$
   (c) $f'(1.05) \approx 2.283$
   (d) $f'(2.3) \approx -19.61$

4. (a) $f'(1) \approx 0.9999$
   (b) $f'(0) \approx 1.997$
   (c) $f'(1.05) \approx 2.282$
   (d) $f'(2.3) \approx -19.66$

5. $\int_0^{\pi} \sin x \, dx \approx 1.999999$

6. $\int_0^{3\pi/2} \cos x \, dx \approx -1.000135$

7. With $h = 0.1$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{1.2}\left[1.8e^{1.8} - 8\left(1.9e^{1.9}\right) + 8(2.1)e^{2.1} - 2.2e^{2.2}\right] = 22.166995.$$

   With $h = 0.05$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{0.6}\left[1.9e^{1.9} - 8\left(1.95e^{1.95}\right) + 8(2.05)e^{2.05} - 2.1e^{2.1}\right] = 22.167157.$$

8. The formula $f'(x_0) = \frac{1}{12h}\left[f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)\right]$ is $O\left(h^3\right)$.

9. Let

$$N_2(h) = N\left(\frac{h}{3}\right) + \left(\frac{N\left(\frac{h}{3}\right) - N(h)}{2}\right) \quad \text{and} \quad N_3(h) = N_2\left(\frac{h}{3}\right) + \left(\frac{N_2\left(\frac{h}{3}\right) - N_2(h)}{8}\right).$$

   Then $N_3(h)$ is an $O(h^3)$ approximation to M.

10. Let $N_2(h) = N\left(\frac{h}{3}\right) + \frac{1}{8}\left(N\left(\frac{h}{3}\right) - N(h)\right)$ and $N_3(h) = N_2\left(\frac{h}{3}\right) + \frac{1}{80}\left(N_2\left(\frac{h}{3}\right) - N_2(h)\right)$. Then $N_3(h)$ is an $O\left(h^6\right)$ approximation to M.

11. Let $N(h) = (1+h)^{1/h}$, $N_2(h) = 2N\left(\frac{h}{2}\right) - N(h)$, $N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{3}(N_2\left(\frac{h}{2}\right) - N_2(h))$.
    (a) $N(0.04) = 2.665836331$, $N(0.02) = 2.691588029$, $N(0.01) = 2.704813829$
    (b) $N_2(0.04) = 2.717339727$, $N_2(0.02) = 2.718039629$. The $O(h^3)$ approximation is $N_3(0.04) = 2.718272931$.
    (c) Yes, since the errors seem proportioned to h for $N(h)$, to h^2 for $N_2(h)$, and to h^3 for $N_3(h)$.

12. (a) We have

$$\lim_{h\to 0}\frac{\ln(2+h)-\ln(2-h)}{h}=\lim_{h\to 0}\frac{1}{2+h}+\frac{1}{2-h}=1,$$

so

$$\lim_{h\to 0}\left(\frac{2+h}{2-h}\right)^{1/h}=\lim_{h\to 0}e^{\frac{1}{h}[\ln(2+h)-\ln(2-h)]}=e^1=e.$$

(b) $N(0.04)=2.718644377221219$, $N(0.02)=2.718372444800607$, $N(0.01)=2.718304481241685$

(c) Let $N_2(h)=2N\left(\frac{h}{2}\right)-N(h)$ and $N_3(h)=N_2\left(\frac{h}{2}\right)+\frac{1}{3}\left[N_2\left(\frac{h}{2}\right)-N_2(h)\right]$. Then $N_2(0.04)=2.718100512379995$, $N_2(0.02)=2.718236517682763$ and $N_3(0.04)=2.718281852783685$. $N_3(0.04)$ is an $O\left(h^3\right)$ approximation satisfying $|e-N_3(0.04)|\le 0.5\times 10^{-7}$.

(d)

$$N(-h)=\left(\frac{2-h}{2+h}\right)^{1/-h}=\left(\frac{2+h}{2-h}\right)^{1/h}=N(h)$$

(e) Let

$$e=N(h)+K_1h+K_2h^2+K_3h^3+\cdots.$$

Replacing h by $-h$ gives

$$e=N(-h)-K_1h+K_2h^2-K_3h^3+\cdots,$$

but $N(-h)=N(h)$, so that

$$e=N(h)-K_1h+K_2h^2-K_3h^3+\cdots.$$

Thus,

$$K_1h+K_3h^3+\cdots=-K_1h-K_3h^3\cdots,$$

and it follows that $K_1=K_3=K_5=\cdots=0$ and

$$e=N(h)+K_2h^2+K_4h^4+\cdots.$$

(f) Let

$$N_2(h)=N\left(\frac{h}{2}\right)+\frac{1}{3}\left(N\left(\frac{h}{2}\right)-N(h)\right)$$

and

$$N_3(h)=N_2\left(\frac{h}{2}\right)+\frac{1}{15}\left(N_2\left(\frac{h}{2}\right)-N_2(h)\right).$$

Then

$$N_2(0.04)=2.718281800660402,\ N_2(0.02)=2.718281826722043$$

and

$$N_3(0.04)=2.718281828459487.$$

$N_3(0.04)$ is an $O\left(h^6\right)$ approximation satisfying

$$|e-N_3(0.04)|\le 0.5\times 10^{-12}.$$

13. (a) We have

$$P_{0,1}(x) = \frac{\left(x - h^2\right) N_1\left(\frac{h}{2}\right)}{\frac{h^2}{4} - h^2} + \frac{\left(x - \frac{h^2}{4}\right) N_1(h)}{h^2 - \frac{h^2}{4}}, \quad \text{so} \quad P_{0,1}(0) = \frac{4N_1\left(\frac{h}{2}\right) - N_1(h)}{3}.$$

Similarly,

$$P_{1,2}(0) = \frac{4N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right)}{3}.$$

(b) We have

$$P_{0,2}(x) = \frac{\left(x - h^4\right) N_2\left(\frac{h}{2}\right)}{\frac{h^4}{16} - h^4} + \frac{\left(x - \frac{h^4}{16}\right) N_2(h)}{h^4 - \frac{h^4}{16}}, \quad \text{so} \quad P_{0,2}(0) = \frac{16N_2\left(\frac{h}{2}\right) - N_2(h)}{15}.$$

14. All the approximations of the form $N_{2i}\left(h/2^j\right)$, for $i = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be upper bounds for M, and all the approximations of the form $N_{2i+1}\left(\frac{h}{2^j}\right)$, for $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, will be lower bounds for M.

15. (a) We have

$$p_4 = 4\sin\left(\frac{\pi}{4}\right) = 4\frac{\sqrt{2}}{2} = 2\sqrt{2} \quad \text{and} \quad P_4 = 4\tan\left(\frac{\pi}{4}\right) = 4 \cdot 1 = 4.$$

(b) We have

$$\frac{2p_k P_k}{p_k + P_k} = \frac{2k\sin\left(\frac{\pi}{k}\right) \cdot k\tan\left(\frac{\pi}{k}\right)}{k\sin\left(\frac{\pi}{k}\right) + \tan\left(\frac{\pi}{k}\right)} = \frac{2k^2\left(\sin\left(\frac{\pi}{k}\right)\right)^2}{k\sin\left(\frac{\pi}{k}\right)\left(1 + \cos\left(\frac{\pi}{k}\right)\right)} = \frac{2k\sin\left(\frac{\pi}{k}\right)}{1 + \cos\left(\frac{\pi}{k}\right)}$$

and

$$P_{2k} = 2k\tan\left(\frac{\pi}{2k}\right) = 2k\frac{\sin\left(\frac{\pi}{2k}\right)}{\cos\left(\frac{\pi}{2k}\right)} = 2k\sqrt{\frac{1 - \cos\left(\frac{\pi}{k}\right)}{1 + \cos\left(\frac{\pi}{k}\right)}}$$

$$= 2k\sqrt{\frac{1 - \cos\left(\frac{\pi}{k}\right)}{1 + \cos\left(\frac{\pi}{k}\right)}} \cdot \sqrt{\frac{1 + \cos\left(\frac{\pi}{k}\right)}{1 + \cos\left(\frac{\pi}{k}\right)}} = 2k\frac{\sqrt{1 - \left(\cos\left(\frac{\pi}{k}\right)\right)^2}}{1 + \cos\left(\frac{\pi}{k}\right)} = \frac{2k\sin\left(\frac{\pi}{k}\right)}{1 + \cos\left(\frac{\pi}{k}\right)}.$$

So $P_{2k} = \dfrac{2p_k P_k}{p_k + P_k}$.

In addition,

$$\sqrt{p_k P_{2k}} = \sqrt{k\sin\left(\frac{\pi}{k}\right) \cdot 2k\tan\left(\frac{\pi}{2k}\right)} = k\sqrt{\left(2\sin\left(\frac{\pi}{2k}\right)\cos\left(\frac{\pi}{2k}\right)\right) \cdot 2\left(\frac{\sin\left(\frac{\pi}{2k}\right)}{\cos\left(\frac{\pi}{2k}\right)}\right)}$$

$$= 2k\sqrt{\left(\sin\left(\frac{\pi}{2k}\right)\right)^2} = 2k\sin\left(\frac{\pi}{2k}\right) = p_{2k}.$$

(c) The polygonal approximations are in the following table.

k	4	8	16	32	64	128	256	512
p_k	$2\sqrt{2}$	3.0614675	3.1214452	3.1365485	3.1403312	3.1412723	3.1415138	3.1415729
P_k	4	3.3137085	3.1825979	3.1517249	3.144184	3.1422236	3.1417504	3.1416321

(d) Values of p_k and P_k are given in the following tables, together with the extrapolation results:

For p_k we have :

2.8284271				
3.0614675	3.1391476			
3.1214452	3.1414377	3.1415904		
3.1365485	3.1415829	3.1415926	3.1415927	
3.1403312	3.1415921	3.1415927	3.1415927	3.1415927

For P_k we have :

4				
3.3137085	3.0849447			
3.1825979	3.1388943	3.1424910		
3.1517249	3.1414339	3.1416032	3.1415891	
3.1441184	3.1415829	3.1415928	3.1415926	3.1415927