1. We have $S(x) = x$ on $[0, 2]$.

2. We have $s(x) = x$ on $[0, 2]$.

3. The equations of the respective free cubic splines are

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following tables.

(a)

i	a_i	b_i	c_i	d_i
0	17.564920	3.13410000	0.00000000	0.00000000

(b)

i	a_i	b_i	c_i	d_i
0	0.22363362	2.17229175	0.00000000	0.00000000

(c)

i	a_i	b_i	c_i	d_i
0	−0.02475000	1.03237500	0.00000000	6.50200000
1	0.33493750	2.25150000	4.87650000	−6.50200000

(d)

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.45508693	0.00000000	−8.9957933
1	−0.28398668	3.18521313	−2.69873800	−0.94630333
2	0.00660095	2.61707643	−2.98262900	9.9420966

4. The equations of the respective free cubic splines are

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

	i	a_i	b_i	c_i	d_i
(a)	0	1.00000000	3.43656000	0.00000000	0.00000000
(b)	0	1.33203000	−1.06249800	0.00000000	0.00000000
(c)	0	−0.29004996	−2.75128630	0.00000000	4.38125000
	1	−0.56079734	−2.61984880	1.31437500	−4.38125000
(d)	0	0.86199480	0.17563785	0.00000000	0.06565093
	1	0.95802009	0.22487604	0.09847639	0.02828072
	2	1.09861230	0.34456298	0.14089747	−0.09393165

5. The following tables show the approximations.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
(a)	8.4	17.87833	17.877146	1.1840×10^{-3}
(b)	0.9	0.4408628	0.44359244	2.7296×10^{-3}
(c)	$-\frac{1}{3}$	0.1774144	0.17451852	2.8959×10^{-3}
(d)	0.25	−0.1315912	−0.13277189	1.1807×10^{-3}

	x	Approximation to $f'(x)$	Actual $f'(x)$	Error
(a)	8.4	3.134100	3.128232	5.86829×10^{-3}
(b)	0.9	2.172292	2.204367	0.0320747
(c)	$-\frac{1}{3}$	1.574208	1.668000	0.093792
(d)	0.25	2.908242	2.907061	1.18057×10^{-3}

6. The following tables show the approximations.

	x	$f(x)$	$s(x)$	Error
(a)	0.43	2.363160694	2.4777208	0.114560106
(b)	0.0	1.000000000	1.066405500	0.066405500
(c)	0.18	−0.5081234644	−0.5079096640	0.0002138004
(d)	0.25	1.189069931	1.192091455	0.003021524

	x	$f'(x)$	$s'(x)$	Error
(a)	0.43	4.726321388	3.436560000	1.289761388
(b)	0.0	−1.000000000	−1.06249800	0.06249800
(c)	0.18	−2.651616829	−2.66716630	0.015549471
(d)	0.25	0.3909913152	0.3973995306	0.0064082154

7. The equations of the respective clamped cubic splines are

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following tables.

(a)

i	a_i	b_i	c_i	d_i
0	17.564920	3.1162560	0.0600867	−0.00202222

(b)

i	a_i	b_i	c_i	d_i
0	0.22363362	2.1691753	0.65914075	−3.2177925

(c)

i	a_i	b_i	c_i	d_i
0	−0.02475000	0.75100000	2.5010000	1.0000000
1	0.33493750	2.18900000	3.2510000	1.0000000

(d)

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.5850208	−2.1498407	−0.49077413
1	−0.28398668	3.1403294	−2.2970730	−0.47458360
2	0.006600950	2.6666773	−2.4394481	−0.44980146

8. The coefficients of the clamped cubic spline interpolation are given in the following table.

	i	a_i	b_i	c_i	d_i
(a)	0	1.00000000	2.00000000	1.74624000	2.25376000
(b)	0	1.33203000	0.43750000	−6.87498800	7.74998400
(c)	0	−0.29004996	−2.80199750	0.97498700	−0.29750000
	1	−0.56079734	−2.61592510	0.88573700	−0.48724000
(d)	0	0.86199480	0.15536240	0.06537475	0.01600323
	1	0.95802009	0.23273957	0.08937959	0.01502024
	2	1.09861230	0.33338433	0.11190995	0.00875797

9. The following tables show the approximations.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
(a)	8.4	17.877144	17.877146	1.6×10^{-6}
(b)	0.9	0.4439248	0.44359244	3.323×10^{-4}
(c)	$-\frac{1}{3}$	0.17451852	0.17451852	0
(d)	0.25	−0.13277221	−0.13277189	3.19×10^{-7}

	x	Approximation to $f'(x)$	Actual $f'(x)$	Error
a	8.4	3.128126	3.128232	1.90×10^{-5}
b	0.9	2.204470	2.204367	1.0296×10^{-4}
c	$-\frac{1}{3}$	1.668000	1.668000	0
d	0.25	2.908242	2.907061	1.18057×10^{-3}

10. The following tables show the approximations.

	x	$f(x)$	$s(x)$	Error
(a)	0.43	2.363160694	2.362069472	0.001091222
(b)	0.0	1.000000000	1.132811750	0.132811750
(c)	0.18	−0.5081234644	−0.4443014992	0.0638219652
(d)	0.25	1.189069931	1.189089597	0.000019666

	x	$f'(x)$	$s'(x)$	Error
(a)	0.43	4.726321388	4.751927072	0.025605684
(b)	0.0	−1.000000000	−1.546872000	0.546872000
(c)	0.18	−2.651616829	−2.325976780	0.325640049
(d)	0.25	0.3909913152	0.3909814244	0.98908×10^{-5}

11. $b = -1$, $c = -3$, $d = 1$

12. $a = 4$, $b = 4$, $c = -1$, $d = \frac{1}{3}$

13. $B = \frac{1}{4}$, $D = \frac{1}{4}$, $b = -\frac{1}{2}$, $d = \frac{1}{4}$

14. $f'(0) = 0$, $f'(2) = 11$

15. The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.0	−0.7573593	0.0	−6.627417
0.25	0.7071068	−2.0	−4.970563	6.627417
0.5	0.0	−3.242641	0.0	6.627417
0.75	−0.7071068	−2.0	4.970563	−6.627417

We have $\int_0^1 S(x)dx = 0.000000$ and $\int_0^1 \cos \pi x \, dx = 0.$

Also, $S'(0.5) = -3.24264$ and $f'(0.5) = -\pi \sin(0.5\pi) = -\pi.$

Finally $S''(0.5) = 0.0$ and $f''(0.5) = -\pi \cos(0.5\pi) = 0.$

16. Assume that the equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$.

The results are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.00000	−0.923601	0	0.620865
0.25	0.778801	−0.807189	0.465649	−0.154017
0.75	0.472367	−0.457052	0.234624	−0.312832

We have

$$\int_0^1 S(x)\,dx = 0.631967, \quad S'(0.5) = -0.603243 \quad \text{and} \quad S''(0.5) = 0.700274.$$

Also,

$$\int_0^1 e^{-x}\,dx = 0.63212056, \quad f'(0.5) = -0.6065307, \quad \text{and} \quad f''(0.5) = 0.6065307.$$

17. Assume that the equation of the spline is

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$.

The coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1	0	−5.193321	2.028118
0.25	0.7071068	−2.216388	−3.672233	4.896310
0.5	0	−3.134447	0	4.896310
0.75	−0.7071068	−2.216388	3.672233	2.028118

We have

$$\int_0^1 s(x)\,dx = 0.000000, \quad s'(0.5) = -3.13445 \quad \text{and} \quad s''(0.5) = 0.000000.$$

Also,

$$\int_0^1 \cos \pi x\,dx = 0, \quad f'(0.5) = -\pi, \quad \text{and} \quad f''(0.5) = 0.$$

18. Assume that the equation of the spline is

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$

The coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.00000	−1.00000	0.499440	−0.154515
0.25	0.778801	−0.779251	0.383555	−0.101580
0.75	0.472367	−0.471881	0.231185	−0.0618174

We have

$$\int_0^1 s(x)\,dx = 0.623078, \quad s'(0.5) = -0.606520, \quad \text{and} \quad s''(0.5) = 0.614740.$$

Also,

$$\int_0^1 e^{-x}\,dx = 0.6321205, \quad f'(0.5) = -0.6065307 \quad \text{and} \quad f''(0.5) = 0.6065307.$$

19. Let $f(x) = a + bx + cx^2 + dx^3$. Clearly, f satisfies properties (a), (c), (d), and (e) of Definition 3.10, and f interpolates itself for any choice of $x_0, \ldots, x_n$. Since (ii) of property (f) in Definition 3.10 holds, f must be its own clamped cubic spline. However, $f''(x) = 2c + 6dx$ can be zero only at $x = -c/3d$. Thus, part (i) of property (f) in Definition 3.10 cannot hold at two values x_0 and x_n. Thus, f cannot be a natural cubic spline.

20. The free cubic spline must be the linear function $L(x)$ through all the data $\{x_i, f(x_i)\}_{i=1}^n$ since $L''(x) = 0$ for all x. So properties (a), (b), (c), (d), (e), (f), (i) of Definition 3.10 would be satisfied.

If f is linear, then f is its own clamped cubic spline. If, for example, f satisfies $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, $f'(0) = 1$, and $f'(2) = 0$, then the data lie on a straight line but the function f is not linear. In that case the spline is

$$s(x) = \begin{cases} x - \frac{1}{4}x^2 + \frac{1}{4}x^3, & 0 \le x \le 1 \\ 1 + \frac{5}{4}(x-1) + \frac{1}{2}(x-1)^2 - \frac{3}{4}(x-1)^3, & 1 \le x \le 2, \end{cases}$$

which is not a linear function.

21. The piecewise linear approximation to f is given by

$$F(x) = \begin{cases} 20(e^{0.1} - 1)x + 1, & \text{for } x \text{ in } [0, 0.05] \\ 20(e^{0.2} - e^{0.1})x + 2e^{0.1} - e^{0.2}, & \text{for } x \text{ in } (0.05, 1]. \end{cases}$$

We have

$$\int_0^{0.1} F(x)\, dx = 0.1107936 \quad \text{and} \quad \int_0^{0.1} f(x)\, dx = 0.1107014.$$

22. We have

$$|f(x) - F(x)| \le \frac{M}{8} \max_{0 \le j \le n-1} |x_{j+1} - x_j|^2,$$

where $M = \max_{a \le x \le b} |f''(x)|$.

Error bounds for Exercise 21 are on $[0, 0.1]$, $|f(x) - F(x)| \le 1.53 \times 10^{-3}$ and

$$\left| \int_0^{0.1} F(x)\, dx - \int_0^{0.1} e^{2x}\, dx \right| \le 1.53 \times 10^{-4}.$$

23. Insert the following before Step 7 in Algorithm 3.4 and Step 8 in Algorithm 3.5:

For $j = 0, 1, \ldots, n-1$ set

$l_1 = b_j$; (*Note that* $l_1 = s'(x_j)$.)
$l_2 = 2c_j$; (*Note that* $l_2 = s''(x_j)$.)
OUTPUT (l_1, l_2)

Set

$l_1 = b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}^2$;(*Note that* $l_1 = s'(x_n)$.)
$l_2 = 2c_{n-1} + 6d_{n-1}h_{n-1}$;(*Note that* $l_2 = s''(x_n)$.)
OUTPUT (l_1, l_2).

24. Before STEP 7 in Algorithm 3.4 and STEP 8 in Algorithm 3.5 insert the following:

Set $I = 0$;
For $j = 0, \ldots, n-1$ set

$$I = a_j h_j + \frac{b_j}{2} h_j^2 + \frac{c_j}{3} h_j^3 + \frac{d_j}{4} h_j^4 + I. \quad \left(\textit{Accumulate} \int_{x_j}^{x_{j+1}} S(x)\, dx. \right)$$

OUTPUT (I).

25. (a) On $[0, 0.05]$, we have $s(x) = 1.000000 + 1.999999x + 1.998302x^2 + 1.401310x^3$, and on $(0.05, 0.1]$, we have $s(x) = 1.105170 + 2.210340(x - 0.05) + 2.208498(x - 0.05)^2 + 1.548758(x - 0.05)^3$.

(b) $\int_0^{0.1} s(x)\, dx = 0.110701$

(c) 1.6×10^{-7}

(d) On $[0, 0.05]$, we have $S(x) = 1 + 2.04811x + 22.12184x^3$, and on $(0.05, 0.1]$, we have $S(x) = 1.105171 + 2.214028(x - 0.05) + 3.318277(x - 0.05)^2 - 22.12184(x - 0.05)^3$. $S(0.02) = 1.041139$ and $S(0.02) = 1.040811$.

26. The five equations are

$$a_0 = f(x_0), \quad a_1 = f(x_1), \quad a_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 = f(x_2),$$

$$a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 = a_1, \quad \text{and} \quad b_0 + 2c_0(x_1 - x_0) = b_1.$$

If $S \in C^2$, then S is a quadratic on $[x_0, x_2]$ and the solution may not be meaningful.

27. We have

$$S(x) = \begin{cases} 2x - x^2, & 0 \le x \le 1 \\ 1 + (x-1)^2, & 1 \le x \le 2 \end{cases}$$

28. (a)

x_i	a_i	b_i	c_i	d_i
1950	151326	2906.50	0.00000	−1.06802
1960	179323	2586.10	−32.0407	1.32212
1970	203302	2341.92	7.62287	−0.94145
1980	226542	2211.94	−20.6208	3.03370
2000	249633	2709.63	70.3902	−2.34634

$S(1940) = 123{,}329$, $S(1975) = 215{,}084$, and $S(2010) = 330{,}922$.

(b) This is much better than using Lagrange interpolation, but still probably not very accurate.

29. The spline has the equation

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	0	75	−0.659292	0.219764
3	225	76.9779	1.31858	−0.153761
5	383	80.4071	0.396018	−0.177237
8	623	77.9978	−1.19912	0.0799115

The spline predicts a position of $s(10) = 774.84$ ft and a speed of $s'(10) = 74.16$ ft/s. To maximize the speed, we find the single critical point of $s'(x)$, and compare the values of $s(x)$ at this point and the endpoints. We find that $\max s'(x) = s'(5.7448) = 80.7$ ft/s $= 55.02$ mi/h. The speed 55 mi/h was first exceeded at approximately 5.5 s.

30. (a) We have

$$t_0 = 0. \quad t_1 = 22.98, \quad t_2 = 47.23, \quad t_3 = 97.49, \quad \text{and} \quad t_4 = 122.66.$$

The coefficients of the spline are given in the following table.

a_i	b_i	c_i	d_i
0.00000000	0.01100783	0.00000000	−0.00000024
0.25000000	0.01062142	−0.00001681	0.00000016
0.50000000	0.01009276	−0.00000499	0.00000004
1.00000000	0.00990986	0.00000135	−0.00000002

This gives

$$s_0(t) = 0.01100783t - 0.00000024t^3,$$
$$s_1(t) = 0.50 + 0.01062142(t - 47.23) - 0.00001681(t - 47.23)^2 + 0.00000016(t - 47.23)^3,$$
$$s_2(t) = 1.00 + 0.00990986(t - 97.49) + 0.00000135(t - 97.49)^2 - 0.00000002(t - 97.49)^3.$$

(b) The predicted time at the three-quarter mile pole is 1 : 12.25, which compares well with the actual time of 1 : 12 : 09.

(c) The starting speed is predicted to be 39.63 mi/h and the speed at the finish line is predicted to be 35.78 mi/h.

31. The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

	Sample 1				Sample 2			
x_i	a_i	b_i	c_i	d_i	a_i	b_i	c_i	d_i
0	6.67	−0.44687	0	0.06176	6.67	1.6629	0	−0.00249
6	17.33	6.2237	1.1118	−0.27099	16.11	1.3943	−0.04477	−0.03251
10	42.67	2.1104	−2.1401	0.28109	18.89	−0.52442	−0.43490	0.05916
13	37.33	−3.1406	0.38974	−0.01411	15.00	−1.5365	0.09756	0.00226
17	30.10	−0.70021	0.22036	−0.02491	10.56	−0.64732	0.12473	−0.01113
20	29.31	−0.05069	−0.00386	0.00016	9.44	−0.19955	0.02453	−0.00102

32. The three clamped splines have equations of the form

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the values of the coefficients are given in the following tables.

Spline 1

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	1	3.0	1.0	−0.347	−0.049	1.0
1	2	3.7	0.447	−0.206	0.027	
2	5	3.9	−0.074	0.033	0.342	
3	6	4.2	1.016	1.058	−0.575	
4	7	5.7	1.409	−0.665	0.156	
5	8	6.6	0.547	−0.196	0.024	
6	10	7.1	0.048	−0.053	−0.003	
7	13	6.7	−0.339	−0.076	0.006	
8	17	4.5				−0.67

Spline 2

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	17	4.5	3.0	−1.101	−0.126	3.0
1	20	7.0	−0.198	0.035	−0.023	
2	23	6.1	−0.609	−0.172	0.280	
3	24	5.6	−0.111	0.669	−0.357	
4	25	5.8	0.154	−0.403	0.088	
5	27	5.2	−0.401	0.126	−2.568	
6	27.7	4.1				−4.0

Spline 3

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	27.7	4.1	0.330	2.262	−3.800	0.33
1	28	4.3	0.661	−1.157	0.296	
2	29	4.1	−0.765	−0.269	−0.065	
3	30	3.0				−1.5

33. The three natural splines have equations of the form

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the values of the coefficients are given in the following tables.

Spline 1

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i
0	1	3.0	0.786	0.0	−0.086
1	2	3.7	0.529	−0.257	0.034
2	5	3.9	−0.086	0.052	0.334
3	6	4.2	1.019	1.053	−0.572
4	7	5.7	1.408	−0.664	0.156
5	8	6.6	0.547	−0.197	0.024
6	10	7.1	0.049	−0.052	−0.003
7	13	6.7	−0.342	−0.078	0.007
8	17	4.5			

Spline 2

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i
0	17	4.5	1.106	0.0	−0.030
1	20	7.0	0.289	−0.272	0.025
2	23	6.1	−0.660	−0.044	0.204
3	24	5.6	−0.137	0.567	−0.230
4	25	5.8	0.306	−0.124	−0.089
5	27	5.2	−1.263	−0.660	0.314
6	27.7	4.1			

Spline 3

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i
0	27.7	4.1	0.749	0.0	−0.910
1	28	4.3	0.503	−0.819	0.116
2	29	4.1	−0.787	−0.470	0.157
3	30	3.0			