1. The interpolation polynomials are as follows.

   (a) $P_1(x) = -0.148878x + 1$; $P_1(0.45) = 0.933005$;
   $|f(0.45) - P_1(0.45)| = 0.032558$;
   $P_2(x) = -0.452592x^2 - 0.0131009x + 1$; $P_2(0.45) = 0.902455$;
   $|f(0.45) - P_2(0.45)| = 0.002008$

   (b) $P_1(x) = 0.467251x + 1$; $P_1(0.45) = 1.210263$;
   $|f(0.45) - P_1(0.45)| = 0.006104$;
   $P_2(x) = -0.0780026x^2 + 0.490652x + 1$; $P_2(0.45) = 1.204998$;
   $|f(0.45) - P_2(0.45)| = 0.000839$

   (c) $P_1(x) = 0.874548x$; $P_1(0.45) = 0.393546$;
   $|f(0.45) - P_1(0.45)| = 0.0212983$;
   $P_2(x) = -0.268961x^2 + 0.955236x$; $P_2(0.45) = 0.375392$;
   $|f(0.45) - P_2(0.45)| = 0.003828$

   (d) $P_1(x) = 1.031121x$; $P_1(0.45) = 0.464004$;
   $|f(0.45) - P_1(0.45)| = 0.019051$;
   $P_2(x) = 0.615092x^2 + 0.846593x$; $P_2(0.45) = 0.505523$;
   $|f(0.45) - P_2(0.45)| = 0.022468$

2. The interpolation polynomials are as follows.

   (a) $P_1(x) = -0.6969992408x + 0.1641422691$; $P_1(1.4) = -0.8116566680$;
   $|f(1.4) - P_1(1.4)| = 0.1393998486$;
   $P_2(x) = 3.552379809x^2 - 10.82128170x + 7.268901887$; $P_2(1.4) = -0.918228067$;
   $|f(1.4) - P_2(1.4)| = 0.0328284496$

   (b) $P_1(x) = 0.6099204008x - 0.1324399760$; $P_1(1.4) = 0.7214485851$;
   $|f(1.4) - P_1(1.4)| = 0.0153577147$;
   $P_2(x) = -3.183202832x^2 + 9.682048472x - 6.498845640$; $P_2(1.4) = 0.816944669$;
   $|f(1.4) - P_2(1.4)| = 0.0801383692$

   (c) $P_1(x) = 0.4012882937x - 0.0622776733$; $P_1(1.4) = 0.4995259379$;
   $|f(1.4) - P_1(1.4)| = 0.0056240404$;
   $P_2(x) = -0.2532041643x^2 + 1.122920162x - 0.5686860021$; $P_2(1.4) = 0.5071220629$;
   $|f(1.4) - P_2(1.4)| = 0.0019720846$

   (d) $P_1(x) = 34.28581783x - 31.92477833$; $P_1(1.4) = 16.07536663$;
   $|f(1.4) - P_1(1.4)| = 1.03071986$;
   $P_2(x) = 26.85344400x^2 - 42.24649756x + 21.78210966$; $P_2(1.4) = 15.26976332$;
   $|f(1.4) - P_2(1.4)| = 0.22511655$

3. Error bounds for the polynomials in Exercise 1 are as follows.

(a) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \leq 0.135$;

For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \leq 0.00397$

(b) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \leq 0.03375$;

For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \leq 0.001898$

(c) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \leq 0.135$;

For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \leq 0.010125$

(d) For $P_1(x)$: $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \leq 0.06779$;

For $P_2(x)$: $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \leq 0.151$

4. Error bounds for the polynomials in Exercise 2 are as follows.

(a) For $P_1(x)$: 0.1480440661; For $P_2(x)$: 0.062012553

(b) For $P_1(x)$: 0.03359789466; There is no bound since the derivative goes to $\infty$.

(c) For $P_1(x)$: 0.004169227026; For $P_2(x)$: 0.006080122747

(d) For $P_1(x)$: 1.471951812; For $P_2(x)$: 1.373821691

5. Interpolation polynomials give the following results.

(a)

| $n$ | $x_0, x_1, \ldots, x_n$ | $P_n(8.4)$ |
|---|---|---|
| 1 | 8.3, 8.6 | 17.87833 |
| 2 | 8.3, 8.6, 8.7 | 17.87716 |
| 3 | 8.3, 8.6, 8.7, 8.1 | 17.87714 |

(b)

| $n$ | $x_0, x_1, \ldots, x_n$ | $P_n(-1/3)$ |
|---|---|---|
| 1 | −0.5, −0.25 | 0.21504167 |
| 2 | −0.5, −0.25, 0.0 | 0.16988889 |
| 3 | −0.5, −0.25, 0.0, −0.75 | 0.17451852 |

(c)

| $n$ | $x_0, x_1, \ldots, x_n$ | $P_n(0.25)$ |
|---|---|---|
| 1 | 0.2, 0.3 | −0.13869287 |
| 2 | 0.2, 0.3, 0.4 | −0.13259734 |
| 3 | 0.2, 0.3, 0.4, 0.1 | −0.13277477 |

(d)

| $n$ | $x_0, x_1, \ldots, x_n$ | $P_n(0.9)$ |
|---|---|---|
| 1 | 0.8, 1.0 | 0.44086280 |
| 2 | 0.8, 1.0, 0.7 | 0.43841352 |
| 3 | 0.8, 1.0, 0.7, 0.6 | 0.44198500 |

6. Interpolation polynomials give the following results.

(a) $P_1(x) = 4.278240000x + 0.579160000$; $P_1(0.43) = 2.418803200$
$|f(0.43) - P_1(0.43)| = 0.055642506$;
$P_2(x) = 5.550800000x^2 + 0.115140000x + 1.273010000$; $P_2(0.43) = 2.348863120$;
$|f(0.43) - P_2(0.43)| = 0.014297574$
$P_3(x) = 2.912106668x^3 + 1.182639999x^2 + 2.117213334x + 1.0$; $P_3(0.43) = 2.360604734$;
$|f(0.43) - P_3(0.43)| = 0.002555960e$

(b) $P_1(x) = -1.062498000x + 1.066405500$; $P_1(0.0) = 1.066405500$
$|f(0.0) - P_1(0.0)| = 0.066405500$;
$P_2(x) = 1.812509334x^2 - 1.062497999x + 0.9531236670$; $P_2(0.0) = 0.9531236670$;
$|f(0.0) - P_2(0.0)| = 0.0468763330$
$P_3(x) = -1.000010667x^3 + 1.312504000x^2 - 0.9999973330x + 0.9843740000$; $P_3(0.0) = 0.9843740000$;
$|f(0.0) - P_3(0.0)| = 0.0156260000$

(c) $P_1(x) = -2.7074748x - 0.01930238$; $P_1(0.18) = -0.506647844$
$|f(0.18) - P_1(0.18)| = 0.0014756204$;
$P_2(x) = 0.8762550000x^2 - 2.970351300x - 0.0017772800$; $P_2(0.18) = -0.5080498520$;
$|f(0.18) - P_2(0.18)| = 0.0000736124$
$P_3(x) = -0.4855333334x^3 + 1.167575000x^2 - 3.023759967x + 0.0011359200$; $P_3(0.18) = -0.5081430745$;
$|f(0.18) - P_3(0.18)| = 0.0000196101$

(d) $P_1(x) = 0.3915288000x + 1.0986123$; $P_1(0.25) = 1.196494500$
$|f(0.25) - P_1(0.25)| = 0.007424569$;
$P_2(x) = 0.1103443800x^2 + 0.3363566100x + 1.098612300$; $P_2(0.25) = 1.189597976$;
$|f(0.25) - P_2(0.25)| = 0.000528045$
$P_3(x) = 0.01414036000x^3 + 0.1103443800x^2 + 0.3328215200x + 1.098612300$; $P_3(0.25) = 1.188935147$;
$|f(0.25) - P_3(0.25)| = 0.000134784$

7. Interpolation polynomials give the following results.

(a)

| $n$ | Actual Error | Error Bound |
|---|---|---|
| 1 | 0.00118 | 0.00120 |
| 2 | $1.367 \times 10^{-5}$ | $1.452 \times 10^{-5}$ |

(b)

| $n$ | Actual Error | Error Bound |
|---|---|---|
| 1 | $4.0523 \times 10^{-2}$ | $4.5153 \times 10^{-2}$ |
| 2 | $4.6296 \times 10^{-3}$ | $4.6296 \times 10^{-3}$ |

(c)

| $n$ | Actual Error | Error Bound |
|---|---|---|
| 1 | $5.9210 \times 10^{-3}$ | $6.0971 \times 10^{-3}$ |
| 2 | $1.7455 \times 10^{-4}$ | $1.8128 \times 10^{-4}$ |

(d)

| $n$ | Actual Error | Error Bound |
|---|---|---|
| 1 | $2.7296 \times 10^{-3}$ | $1.4080 \times 10^{-2}$ |
| 2 | $5.1789 \times 10^{-3}$ | $9.2215 \times 10^{-3}$ |

8. Error bounds when $n = 1$ and $n = 2$ are as follows.

(a) 0.06850070205 and 0.02409356045

(b) 0.2656250000 and 0.09375000000

(c) 0.001552099938 and 0.0001109161632

(d) 0.007740087700 and 0.0007457301283

9. We have $y = 4.25$.

10. The largest value is $x_1 = 0.872677996$.

11. We have $P_3(1.09) \approx 0.2826$. The actual error is $4.3 \times 10^{-5}$, and an error bound is $7.4 \times 10^{-6}$. The discrepancy is due to the fact that the data are given to only four decimal places, and only four-digit arithmetic is used.

12. The approximation is $\cos 0.75 \approx 0.7313$. The actual error is 0.0004, and an error bound is $2.7 \times 10^{-8}$. The discrepancy is due to the fact that the data are given only to four decimal places and four digit arithmetic is used.

13. (a) $P_2(x) = -11.22388889x^2 + 3.810500000x + 1$.
An error bound is 0.11371294.

(b) $P_2(x) = -0.1306344167x^2 + 0.8969979335x - 0.63249693$.
An error bound is $9.45762 \times 10^{-4}$.

(c) $P_3(x) = 0.1970056667x^3 - 1.06259055x^2 + 2.532453189x - 1.666868305$.
An error bound is $10^{-4}$.

(d) $P_3(x) = -0.07932x^3 - 0.545506x^2 + 1.0065992x + 1$.
An error bound is $1.591376 \times 10^{-3}$.

14. (a) 1.32436

(b) 2.18350

(c) $1.15277, 2.01191$

(d) Parts (a) and (b) are better due to the spacing of the nodes.

15. Using 10 digits gives $P_3(x) = 1.302637066x^3 - 3.511333118x^2 + 4.071141936x - 1.670043560$, $P_3(1.09) = 0.282639050$, and $|f(1.09) - P_3(1.09)| = 3.8646 \times 10^{-6}$.

16. 0.7317039560

17. The largest possible step size is 0.004291932, so 0.004 would be a reasonable choice.

18. (a) We have

$$-0.02158783335x^5 + 213.3039543x^4 - 843021.8192x^3 + 166586109x^2 \\ - 1.645882427 \times 10^{12}x + 6.504424241 \times 10^{14},$$

which gives $P(1940) = 642397$, $P(1975) = 230863$, and $P(2020) = -1754557$.

(b) The actual population in 1940 was approximately 132,165,000 as compared to the approximation of 642,397,000, does not lend confidence to the other approximations. Further, the 1975 estimate is also too large and the estimate for 2020 is negative. This is a very poor approximating polynomial.

19. (a) Sample 1: $P_6(x) = 6.67 - 42.6434x + 16.1427x^2 - 2.09464x^3 + 0.126902x^4 - 0.00367168x^5 + 0.0000409458x^6$;

Sample 2: $P_6(x) = 6.67 - 5.67821x + 2.91281x^2 - 0.413799x^3 + 0.0258413x^4 - 0.000752546x^5 + 0.00000836160x^6$

(b) Sample 1: 42.71 mg; Sample 2: 19.42 mg

20. (a) We have

| $x$ | erf($x$) |
|---|---|
| 0.0 | 0 |
| 0.2 | 0.2227 |
| 0.4 | 0.4284 |
| 0.6 | 0.6039 |
| 0.8 | 0.7421 |
| 1.0 | 0.8427 |

(b) Linear interpolation with $x_0 = 0.2$ and $x_1 = 0.4$ gives $\text{erf}(\frac{1}{3}) \approx 0.3598$, and quadratic interpolation with $x_0 = 0.2$, $x_1 = 0.4$, and $x_2 = 0.6$ gives $\text{erf}(\frac{1}{3}) \approx 0.3632$. Since $\text{erf}(\frac{1}{3}) \approx 0.3626$, quadratic interpolation is more accurate.

21. Since $g(x) = g(x_0) = 0$, there exists a number $\xi_1$ between $x$ and $x_0$, for which $g'(\xi_1) = 0$. Also, $g'(x_0) = 0$, so there exists a number $\xi_2$ between $x_0$ and $\xi_1$, for which $g''(\xi_2) = 0$. The process is continued by induction to show that a number $\xi_{n+1}$ between $x_0$ and $\xi_n$ exists with $g^{(n+1)}(\xi_{n+1}) = 0$. The error formula for Taylor polynomials follows.

22. Since $g'\left(\left(j + \frac{1}{2}\right)h\right) = 0$,

$$\max|g(x)| = \max\left\{|g(jh)|, \left|g\left(\left(j + \frac{1}{2}\right)h\right)\right|, |g((j+1)h)|\right\} = \max\left(0, \frac{h^2}{4}\right),$$

so $|g(x)| \leq h^2/4$.

23. (a) For (i) we have

$$B_3(x) = \sum_{k=0}^{3} \binom{3}{k} \left(\frac{k}{3}\right) x^k(1-x)^{3-k} = 0 + \binom{3}{1} \left(\frac{1}{3}\right) x(1-x)^2 + \binom{3}{2} \left(\frac{2}{3}\right) x^2(1-x) + \binom{3}{3} \left(\frac{3}{3}\right) x^3$$
$$= 3 \cdot \left(\frac{1}{3}\right) x(1-2x+x^2) + 3 \cdot \left(\frac{2}{3}\right) (x^2 - x^3) + x^3 = x - 2x^2 + x^3 + 2x^2 - 2x^3 + x^3 = x$$

For (ii) we have

$$B_3(x) = \sum_{k=0}^{3} \binom{3}{k} x^k(1-x)^{3-k} = \binom{3}{0}(1-x)^3 + \binom{3}{1}x(1-x)^2 + \binom{3}{2}x^2(1-x) + \binom{3}{3}x^3$$
$$= (1-x)^3 + 3 \cdot x(1-2x+x^2) + 3 \cdot (x^2 - x^3) + x^3$$
$$= (1 - 3x + 3x^2 - x^3) + (3x - 6x^2 + 3x^3) + (3x^2 - 3x^3) + x^3 = 1$$

(b) We have

$$\frac{k}{n} \cdot \binom{n}{k} = \frac{k}{n} \cdot \frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \binom{n-1}{k-1}.$$

(c) First we need to show that when $f(x) = 1$ and $n \geq 0$ we have $B_n(x) = 1$, and when $f(x) = x$ and $n \geq 1$ we have $B_n(x) = x$. The first of these follows directly from the Binomial Theorem. For the second we will use mathematical induction on $n$.

When $f(x) = x$ and $n = 1$ we have

$$B_1(x) = \sum_{k=0}^{1} \binom{1}{k} \left(\frac{k}{1}\right) x^k(1-x)^{1-k} = 0 + 1 \cdot 1 \cdot x = x.$$

Now assume that for an arbitrary positive integer $n$ we have $B_n(x) = x$. From part (b) we have

$$\begin{aligned} B_{n+1}(x) &= \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{k}{n+1}\right) x^k (1-x)^{n+1-k} \\ &= \sum_{k=1}^{n+1} \binom{n+1}{k} \left(\frac{k}{n+1}\right) x^k (1-x)^{n+1-k} \\ &= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k (1-x)^{n+1-k} \end{aligned}$$

Let $j = k-1$.

$$= x \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} = x \cdot 1 = x.$$

This establishes the fact that when $f(x) = x$, $B_n(x) = x$ for all $n \geq 1$.
Now consider the case of $f(x) = x^2$. For this function the $n$th Bernstein polynomial is

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^2 x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \left(\binom{n}{k} \frac{k}{n}\right) \left(\frac{k}{n}\right) x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\frac{k}{n}\right) x^k (1-x)^{n-k} \end{aligned}$$

Let $j = k-1$. Then

$$\begin{aligned} B_n(x) &= \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{j+1}{n}\right) x^{j+1} (1-x)^{n-(j+1)} \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{j}{n}\right) x^{j+1} (1-x)^{n-(j+1)} + \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{1}{n}\right) x^{j+1} (1-x)^{n-(j+1)} \\ &= \left(\frac{n-1}{n}\right) x \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{j}{n-1}\right) x^j (1-x)^{n-1-j} + \left(\frac{1}{n}\right) x \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} \\ &= \left(\frac{n-1}{n}\right) x \cdot x + \left(\frac{1}{n}\right) x \cdot 1 = \frac{n-1}{n} x^2 + \frac{1}{n} x. \end{aligned}$$

(d) $n \geq 250{,}000$