1. (a) For $p_0 = 0.5$, we have $p_{13} = 0.567135$.
   (b) For $p_0 = -1.5$, we have $p_{23} = -1.414325$.
   (c) For $p_0 = 0.5$, we have $p_{22} = 0.641166$.
   (d) For $p_0 = -0.5$, we have $p_{23} = -0.183274$.

2. (a) For $p_0 = 0.5$, we have $p_{15} = 0.739076589$.
   (b) For $p_0 = -2.5$, we have $p_9 = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
   (d) For $p_0 = 4.0$, we have $p_{44} = 3.37354190$.

3. Modified Newton's method in Eq. (2.11) gives the following:
   (a) For $p_0 = 0.5$, we have $p_3 = 0.567143$.
   (b) For $p_0 = -1.5$, we have $p_2 = -1.414158$.
   (c) For $p_0 = 0.5$, we have $p_3 = 0.641274$.
   (d) For $p_0 = -0.5$, we have $p_5 = -0.183319$.

4. (a) For $p_0 = 0.5$, we have $p_4 = 0.739087439$.
   (b) For $p_0 = -2.5$, we have $p_{53} = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
   (d) For $p_0 = 4.0$, we have $p_3 = -3.72957639$.

5. Newton's method with $p_0 = -0.5$ gives $p_{13} = -0.169607$. Modified Newton's method in Eq. (2.11) with $p_0 = -0.5$ gives $p_{11} = -0.169607$.

6. (a) Since
$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1,$$
   we have linear convergence. To have $|p_n - p| < 5\times 10^{-2}$, we need $n \geq 20$.
   (b) Since
$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 1,$$
   we have linear convergence. To have $|p_n - p| < 5\times 10^{-2}$, we need $n \geq 5$.

7. (a) For $k > 0$,
$$\lim_{n\to\infty}\frac{|p_{n+1}-0|}{|p_n-0|} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^k = 1,$$
   so the convergence is linear.
   (b) We need to have $N > 10^{m/k}$.

8. (a) Since

$$\lim_{n\to\infty}\frac{|p_{n+1}-0|}{|p_n-0|^2}=\lim_{n\to\infty}\frac{10^{-2^{n+1}}}{(10^{-2^n})^2}=\lim_{n\to\infty}\frac{10^{-2^{n+1}}}{10^{-2^{n+1}}}=1,$$

the sequence is quadratically convergent.

(b) We have

$$\lim_{n\to\infty}\frac{|p_{n+1}-0|}{|p_n-0|^2}=\lim_{n\to\infty}\frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2}=\lim_{n\to\infty}\frac{10^{-(n+1)^k}}{10^{-2n^k}}$$
$$=\lim_{n\to\infty}10^{2n^k-(n+1)^k}=\lim_{n\to\infty}10^{n^k(2-(\frac{n+1}{n})^k)}=\infty,$$

so the sequence $p_n=10^{-n^k}$ does not converge quadratically.

9. Typical examples are

(a) $p_n=10^{-3^n}$

(b) $p_n=10^{-\alpha^n}$

10. Suppose $f(x)=(x-p)^m q(x)$. Since

$$g(x)=x-\frac{m(x-p)q(x)}{mq(x)+(x-p)q'(x)},$$

we have $g'(p)=0$.

11. This follows from the fact that

$$\lim_{n\to\infty}\frac{\left|\frac{b-a}{2^{n+1}}\right|}{\left|\frac{b-a}{2^n}\right|}=\frac{1}{2}.$$

12. If f has a zero of multiplicity m at p, then f can be written as

$$f(x) = (x-p)^m q(x),$$

for $x \neq p$, where

$$\lim_{x\to p} q(x) \neq 0.$$

Thus,

$$f'(x) = m(x-p)^{m-1}q(x) + (x-p)^m q'(x)$$

and $f'(p) = 0$. Also,

$$f''(x) = m(m-1)(x-p)^{m-2}q(x) + 2m(x-p)^{m-1}q'(x) + (x-p)^m q''(x)$$

and $f''(p) = 0$. In general, for $k \leq m$,

$$f^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j(x-p)^m}{dx^j} q^{(k-j)}(x) = \sum_{j=0}^{k} \binom{k}{j} m(m-1)\cdots(m-j+1)(x-p)^{m-j}q^{(k-j)}(x).$$

Thus, for $0 \leq k \leq m-1$, we have $f^{(k)}(p) = 0$, but $f^{(m)}(p) = m! \lim_{x\to p} q(x) \neq 0$.

Conversely, suppose that

$$f(p) = f'(p) = \ldots = f^{(m-1)}(p) = 0 \quad \text{and} \quad f^{(m)}(p) \neq 0.$$

Consider the $(m-1)$th Taylor polynomial of f expanded about p:

$$\begin{aligned} f(x) =& f(p) + f'(p)(x-p) + \ldots + \frac{f^{(m-1)}(p)(x-p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x-p)^m}{m!} \\ =&(x-p)^m \frac{f^{(m)}(\xi(x))}{m!}, \end{aligned}$$

where $\xi(x)$ is between x and p.

Since $f^{(m)}$ is continuous, let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}.$$

Then $f(x) = (x-p)^m q(x)$ and

$$\lim_{x\to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0.$$

Hence f has a zero of multiplicity m at p.

13. If

$$\frac{|p_{n+1}-p|}{|p_n-p|^3} = 0.75 \quad \text{and} \quad |p_0-p| = 0.5, \quad \text{then} \quad |p_n-p| = (0.75)^{(3^n-1)/2}|p_0-p|^{3^n}.$$

To have $|p_n - p| \leq 10^{-8}$ requires that $n \geq 3$.

14. Let $e_n = p_n - p$. If

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0,$$

then for sufficiently large values of n, $|e_{n+1}| \approx \lambda |e_n|^\alpha$. Thus,

$$|e_n| \approx \lambda |e_{n-1}|^\alpha \quad \text{and} \quad |e_{n-1}| \approx \lambda^{-1/\alpha} |e_n|^{1/\alpha}.$$

Using the hypothesis gives

$$\lambda |e_n|^\alpha \approx |e_{n+1}| \approx C|e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha}, \quad \text{so} \quad |e_n|^\alpha \approx C\lambda^{-1/\alpha-1}|e_n|^{1+1/\alpha}.$$

Since the powers of $|e_n|$ must agree,

$$\alpha = 1 + 1/\alpha \ \text{ and } \ \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62.$$

The number α is the *golden ratio* that appeared in Exercise 16 of section 1.3.